Unit-1 Sequences of Real Numbers

1. INTRODUCTION (SEQUENCES OF REAL NUMBERS)

By a sequence, we mean an arrangement of numbers in a definite order according to some rule. We denote the terms of a sequence by a_1, a_2, a_3, ..., etc., the subscript denotes the position of the term.

In view of the above a sequence in the set X can be regarded as a mapping or a function f : N → X defined by

$$f(n) = a_n \ \forall \ n \in N$$

Domain of f is a set of natural numbers or some subset of it denoting the position of term. If its range denoting the value of terms is a subset of R real numbers then it is called a **real sequence**.

A sequence is either finite or infinite depending upon the number of terms in a sequence.

We should not expect that its terms will be necessarily given by a specific formula.

However, we expect a theoretical scheme or rule for generating the terms.

Sequence following certain patterns are more often called **progressions**. In progressions, we note that each term except the first progresses in a definite manner.

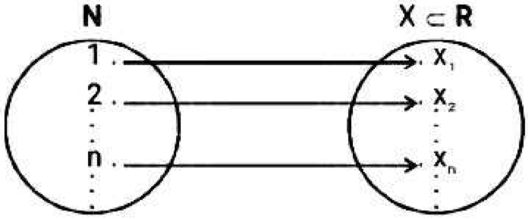
A sequence of real numbers is a function whose domain is a set of the form $\{n \in \mathbb{Z} \mid n \geq m\}$ where m is usually 0 or 1. Thus, a sequence is a function $f : \mathbb{N} \to \mathbb{R}$. Thus a sequence can be denoted by f(m), f(m + 1), f(m + 2), ... Usually, we will denote such a sequence by $\{a_i\}_{i=m}^{\infty}$ or $\{a_m, a_{m+1}, a_{m+2}, \ldots\}$, where $a_i = f(i)$. If m = 1, we may use the notation $\{a_n\}_{n \in N}$.

Definition : Every function defined from the set **N** of natural numbers to a non-empty set X is called a sequence.

2. REAL SEQUENCE

Every function defined from the set **N** of natural numbers to a non-empty subset X of the set of real numbers **R** is called a real sequence. denoted by $f : N \to R$.

Thus the real sequence f is set of all ordered pairs $\{n, f(n)\} \mid \{n = 1, 2, 3, ...\}$ i.e., set of all pairs $(n, f(n))$ with n a positive integer.



If $f : \mathbf{N} \to \mathbf{R}$ is a sequence, then for each $n \in \mathbf{N}$, $f(n)$ is a real number. It is conventional to write $f(n)$ as f_n.

Notations: Since the domain of a sequence is always the same (the set of positive integers) a sequence may be written as $\{f(n)\}$ instead of $\{n, f(n)\}$.

Example. The sequence {1, 1/2, 1/3, 1/4, 1/5, ...} is written as $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$. This sequence can be thought of as an ordinary function $f(n) = 1/n$.

Example. Consider the sequence given by $a_n = (-1)^n$ for $n \geq 0$. The terms of the sequence look like, {1, –1, 1, –1, 1, –1, ...}. Note that the function has domain $\mathbb{N}$ but the range is {–1, 1}.

Example. Consider the sequence $a_n = \cos\left\{\frac{n\pi}{3}\right\}$, $n \in \mathbb{N}$. The first terms in the sequence is $\cos\frac{\pi}{3} = \cos 60° = \frac{1}{2}$ and the sequence looks like

$$\{\frac{1}{2},-\frac{1}{2},-1,-\frac{1}{2},\frac{1}{2},1,\frac{1}{2},-\frac{1}{2},-1,-\frac{1}{2},\frac{1}{2},1,\frac{1}{2},-\frac{1}{2},-1,...\}.$$

Note that the function takes on only a finite number of values, but the sequence has an infinite number of elements.

Example. If $a_n = n^{1/n}$, $n \in \mathbb{N}$, the sequence is

$$1,\sqrt{2},3^{1/3},4^{1/4},...$$

Example. Consider the sequence $b_n = \left(1+\frac{1}{n}\right)^n$, $n \in \mathbb{N}$. This is the sequence

$$2,\left(\frac{3}{2}\right)^3,\left(\frac{4}{3}\right)^3,\left(\frac{5}{4}\right)^4,....$$

Representation of A Sequence

The real numbers $x_1, x_2, ..., x_n, ...$ are called the terms or elements of the sequence. x_1 is called the first term, x_2 the second term, ..., x_n the nth term of the sequence $\langle x_n \rangle$. It is denoted by $\langle x_1, x_2, x_3, ..., x_n, ... \rangle$ or $\langle x_n \rangle$ or $\{x_n\}$.

Example.

(i). $\left\langle \frac{1}{n} \right\rangle = \left\langle 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots \right\rangle$.

(ii). $\left\langle 1+(-1)^n \right\rangle = \langle 0, 2, 0, 2, \ldots \rangle$.

(iii). $\left\langle n^2 \right\rangle = \langle 1, 4, 9, 16, \ldots \rangle$.

(iv). $\left\langle (-1)^n \right\rangle = \langle -1, 1, -1, 1, \ldots \rangle$.

(v). A sequence $\langle a_n \rangle$ may be defined by a recursion formula :

$$a_{n+1} = \sqrt{2a_n},\ a_1 = 1.$$

$$a_2 = \sqrt{2a_1} = \sqrt{2.1} = \sqrt{2}$$

$$a_3 = \sqrt{2a_2} = \sqrt{2.\sqrt{2}}$$

$$\vdots$$

Here the terms of the sequence are 1, $\sqrt{2}, \sqrt{2\sqrt{2}}, \ldots$

3. RANGE OF A SEQUENCE

The set of all the distinct elements of a sequence is called the range set of the given sequence.

Example.

The range sets of the sequences given in **Example** are respectively

(i) $\left\{1, \frac{1}{2}, \frac{1}{3}, \ldots\right\}$ (ii) $\{0, 2\}$

(iii) $\{1, 4, 9, \ldots\}$ (iv) $\{-1, 1\}$

(v) $\{1, \sqrt{2}, \sqrt{2\sqrt{2}}, \ldots\}$

Note : The range set of a sequence may be finite or infinite but the sequence has always an infinite number of elements.

Particular Sequences

(a) **Constant Sequence :** If the n^{th} term of the sequence is constant i.e. $a_n = c \in R, \forall n$, then the sequence obtained $\langle c, c, c, \ldots \rangle$ is known as constant sequence.

Example. : $\langle 5, 5, 5, 5, \ldots \rangle = \langle 5 \rangle$

$\langle 0, 0, 0, 0, \ldots \rangle = \langle 0 \rangle$

(b) **Identity Sequence :** If the n^{th} term of the sequence is $a_n = n$, then the obtained sequence is called Identity sequence.

Example. : $\langle 1, 2, 3, \ldots\ldots \rangle$

Equal Sequence

Two sequences $<a_n>$ and $<b_n>$ are equal, if their n^{th} terms are equal.

i.e. $a_n = b_n, \forall n \in \mathbf{N}$

4. BOUNDED AND MONOTONE SEQUENCES

Bounded Sequence

Bounded above : A sequence $\langle a_n \rangle$ is said to be **bound above,** if $\exists$ a real number K s.t. $a_n \le K \ \forall \ n \in \mathbf{N}$. K is called an **upper bound** of the sequence $\langle a_n \rangle$.

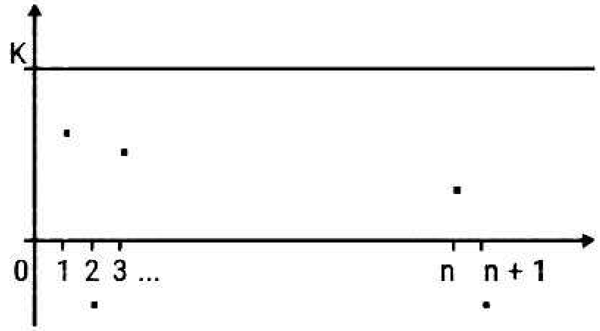


Figure. Bounded above sequence

Bounded below : A sequence $\langle a_n \rangle$ is said to be **bounded below,** if $\exists$ a real number k s.t. $k \le a_n \ \forall \ n \in \mathbf{N}$. k is called a **lower bound** of the sequence $\langle a_n \rangle$.

Bounded : A sequence $\langle a_n \rangle$ is said to be **bounded,** if it is both bounded above and bounded below.

Equivalently, $\langle a_n \rangle$ is **bounded,** if there exist two real numbers k and K such that $k \le a_n \le K \ \forall \ n \in \mathbf{N}$.

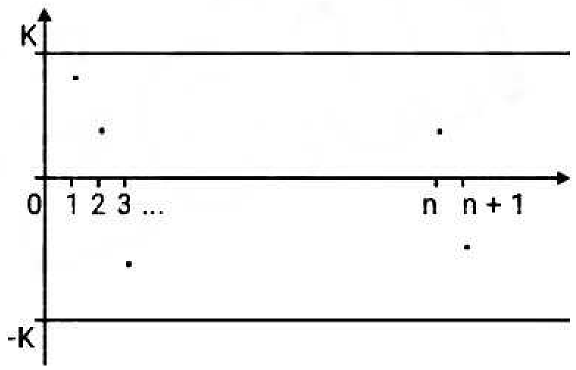


Figure. Bounded sequence

Examples

1. $\left\langle \frac{1}{n} \right\rangle = \left\langle 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots \right\rangle$ is bounded, as $0 < \frac{1}{n} \le 1 \ \forall \ n \in \mathbf{N}$.
2. $\langle 1 + (-1)^n \rangle = \langle 0, 2, 0, 2, \ldots \rangle$ is bounded.

Supremum and Infimum of a Sequence

The minimum value of the upper bounds of sequence is known as Supremum or least upper bound (lub) of a sequence.

The greatest of the lower bounds of a sequence is known as the Infimum or greatest lower bound (glb) of a sequence.

Examples

1. The sequence $\{n^2\}$ is bounded below: $n^2 > 0 \ \forall \ n \in \mathbf{N}$ but not bounded above;
2. The sequence $\{-n\}$ is bounded above: $-n < 0 \ \forall \ n \in \mathbf{N}$ but not bounded below;
3. The sequence $\{(-1)^n + 1\}$ is bounded : $|(-1)^n + 1| \le 2 \ \forall \ n \in \mathbf{N}$.

Infinitely large sequences represent an important subset of unbounded sequences.

Definition : A sequence $\{a_n\}$ is called infinitely large if $\forall K \in \mathbf{R}\ \exists\ n_K \in \mathbf{N}$ such that

$|a_n| > K\ \forall\ n \geq n_K$.

As an example, we show that the sequence $\{(-1)^n n^3\}$ is infinitely large.

Indeed, for any number K, we can find n_k such that $|(-1)^n n^3| > K\ \forall n \geq n_K$.

To this end, we solve the inequality $n^3 > K$, and $n > \sqrt[3]{K}$. t

Let $n_k = [\sqrt[3]{K}] + 1$, where [c] is the integer part of c.

Then for $n \geq n_K$ we obtain.

$$n \geq n_k > \sqrt[3]{K} \Rightarrow n^3 > K \Rightarrow \left|(-1)^n\ n^3\right| > K.$$

From above Definition it follows that any infinitely large sequence is unbounded. However, the converse is not true: there exist unbounded sequences that are not infinitely large.

For example, such is the sequence $\{(1 - (-1)^n)n\}$.

Definition. A sequence $\{a_n\}$ is called infinitely small if

$$\lim_{n\to\infty} a_n = 0,$$

that is for any $\varepsilon > 0$ there exists n_ε such that

$$|a_n| < \varepsilon\ \forall\ n \geq n_\varepsilon.$$

For example, the sequence $\{q^n\}$ for $|q| < 1$ is infinitely small. Indeed, for any $\varepsilon > 0$ let us find n_ε such that $|q^n| < \varepsilon\ \forall\ n \geq n_\varepsilon$. To this end, we solve the inequality $|q^n| < \varepsilon$, assuming that $0 < \varepsilon < 1$ (for $\varepsilon \geq 1$, this inequality is clearly true for any $n \in \mathbf{N}$):

$$n \ln(|q|) < \ln \varepsilon \quad \Rightarrow \quad n > \frac{\ln\varepsilon}{\ln|q|}, \quad n_\varepsilon = \left\lfloor \frac{\ln \varepsilon}{\ln |q|} \right\rfloor + 1,$$

where $\ln\varepsilon < 0$, and $\ln |q| < 0$, since $\varepsilon < 1$, and $|q| < 1$. Thus, for $n \geq n_\varepsilon$ we have

$$n \geq n_\varepsilon > \frac{\ln \varepsilon}{\ln |q|} < \ln \varepsilon \Rightarrow |q|^n < \varepsilon.$$

The fact that $\{a_n\}$ is not infinitely small means the following there exists $\varepsilon_0 > 0$ such that for any $n \in \mathbf{N}$ there exists $k_n > n$ with $\left|a_{k_n}\right| > \varepsilon_0$.

Unbounded Sequence

Sequence which is either unbounded above or below is called an unbounded sequence.

Monotone Sequences

Monotonically Increasing : A sequence $\langle a_n\rangle = \langle a_1, a_2, a_3, ..., a_n, ...\rangle$ is said to be monotonically increasing, if $a_{n+1} \geq a_n\ \forall\ n \in \mathbf{N}$.

Monotonically Decreasing : A sequence $\langle a_n\rangle = \langle a_1, a_2, a_3, ..., a_n, ...\rangle$ is said to be monotonically decreasing, if $a_{n+1} \leq a_n\ \forall\ n \in \mathbf{N}$.

Monotonic : A sequence $\langle a_n\rangle$ is said to be **monotonic (or monotone)** if it is either monotonically increasing or monotonically decreasing.

Strictly monotonic sequence : either strictly increasing or strictly decreasing.

In particular, monotonically increasing is the same as increasing, strictly monotonically increasing the same as strictly increasing.

Examples

1. The sequence $\langle 2^n \rangle = \langle 2, 4, 8, 16, \ldots \rangle$ is monotonically increasing.

   Since $a_1 = 2 < a_2 = 4 < a_3 = 8 < \ldots\ldots.$

2. The sequence $\left\langle \frac{1}{n} \right\rangle = \left\langle 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots \right\rangle$ is monotonically decreasing.

   Since $a_1 = 1 > a_2 = \frac{1}{2} > a_3 = \frac{1}{3} > \ldots\ldots.$

3. The sequence $\left\{ \frac{1}{n^2} \right\} = 1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \frac{1}{25}, \cdots$ is a bounded monotone decreasing sequence. Its upper bound is greater than or equal to 1, and the lower bound is any non-positive number. The least upper bound is number one, and the greatest lower bound is zero, that is,

$$0 < \frac{1}{n^2} \le 1$$

   for each natural number n.

4. The sequence $\left\{ \frac{n}{n+1} \right\} = \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \ldots$ is a bounded monotone increasing sequence.

   The least upper bound is number one, and the greatest lower bound is 1/2, that is,

$$\frac{1}{2} \le \frac{n}{n+1} < 1$$

   for each natural number n.

5. The sequence $\left\{ \frac{n^2}{n+1} \right\} = \frac{1}{2}, \frac{4}{3}, \frac{9}{4}, \frac{16}{5}, \ldots$ is an unbounded sequence, because it does not have a finite upper bound.

6. Monotone increasing sequences :

$$\{n\} = 1, 2, 3, \ldots$$

$$\left\{ \frac{n}{n+1} \right\} = \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \ldots$$

$$\left\{ \frac{n^2}{n+1} \right\} = \frac{1}{2}, \frac{4}{3}, \frac{9}{4}, \frac{16}{5}, \ldots$$

7. Monotone decreasing sequences :

$$\left\{ \frac{1}{n} \right\} = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$$

$$\left\{ \frac{1}{n^2} \right\} = 1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \frac{1}{25}, \ldots$$

$$\left\{ \frac{1}{2^n} \right\} = \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \ldots$$

Eventual nature of a sequence

Definition. A sequence $\{a_n\}$ of real numbers is called non-decreasing if $a_n \leq a_{n+1}$ for all n, and it is called non-increasing if $a_n \geq a_{n+1}$ for all n. It is called strictly increasing if $a_n < a_{n+1}$ for all n, and strictly decreasing if $a_n > a_{n+1}$ for all n.

A sequence $\{a_n\}$ of real numbers is called eventually non-decreasing if there exists a natural number **N** such that $a_n \leq a_{n+1}$ for all $n \geq$ **N**, and it is called eventually non-increasing if there exists a natural number **N** such that $a_n \geq a_{n+1}$ for all $n \geq$ **N**. We make analogous definitions of "eventually strictly increasing" and "eventually strictly decreasing."

5. LIMIT POINT OF A SEQUENCE

Let $\{a_n\}$ be any sequence and $\alpha \in \mathbb{R}$ we say α is limit point of $\{a_n\}$ if every neighbourhood of α contains infinite members of the sequence $\{a_n\}$.

i.e. for any $\delta > 0$

$a_n \in (\alpha - \delta, \alpha + \delta)$ for infinite values of n.

Ex. (i). $a_n = (-1)^n$

$\alpha = -1$

for any $\delta > 0$

$a_n \in (-1 - \delta, -1 + \delta)\ \forall\ n = 2k - 1, k = 1, 2, ...$

$\therefore$ -1 is a limit point.

$\alpha_2 = 1.$

for any $\delta > 0$

$a_n \in (1 - \delta, 1 + \delta)\ \forall\ n = 2k, k = 1, 2, ...$

$\therefore$ 1 is a limit point.

$\{a_n\}$ has two limit points $\{-1, 1\}$.

(ii). $a_n = \begin{cases} 2; & n = 1 \text{ or prime.} \\ p; & p|n \text{ and } p \text{ is the least prime doing so.} \end{cases}$

$\alpha = 2$

for any $\delta > 0$

$a_n \in (2 - \delta, 2 + \delta)\ \forall\ n = 1$ or prime

$\therefore$ 2 is a limit point.

Let p be any prime.

for any $\delta > 0$

$a_n \in (p - \delta, p + \delta)\ \forall\ n = p^k, k = 1, 2, ...$

$\therefore$ p is the limit point of a_n.

Hence, every prime no. is a limit point of $\{a_n\}$.

As set of prime no. are infinite

$\therefore$ $\{a_n\}$ has infinite no. of limit points.

Theorem : Every limit point of the range set of a sequence is limit point of a sequence.

Solution : Let S be range set of sequence $\{a_n\}$.

i.e. S = range of $\{a_n\}$

Let $\alpha \in S'$

for $\varepsilon > 0$

$(\alpha - \varepsilon, \alpha + \varepsilon) \cap S\backslash\{\alpha\}$ has infinite no. of points

Let $q \in (\alpha - \varepsilon, \alpha + \varepsilon) \cap S\backslash\{\alpha\}$

$\Rightarrow$ $q \in (\alpha - \varepsilon, \alpha + \varepsilon)$ and $q \in S$

As $q \in S$

$\Rightarrow$ $q = a_k$ for some $k \in \mathbb{N}$

So $a_k \in (\alpha - \varepsilon, \alpha + \varepsilon)$

Hence, $(\alpha - \varepsilon, \alpha + \varepsilon)$ contains infinite no. of terms of sequence

$\therefore$ α is limit point of sequence $\{a_n\}$ **Proved**

Remark : A real no. is a limit point of sequence $\Leftrightarrow$ it appears in the sequence infinite many times or it is the limit point of the range set.

6. BOLZANO-WEIERSTRASS THEOREM

Theorem : Every bounded sequence has a limit point.

Proof : Let $\langle a_n \rangle$ be a bounded sequence.

Let $S = \{a_n : n \in \mathbf{N}\}$ be its range.

Since the sequence is bounded, therefore, its range S is also bounded.

Case I. Let S be a finite set.

Then there must exist at least one element $\alpha \in S$ such that

$a_n = \alpha$ for an infinite number of values of n.

For any $\varepsilon > 0$, the nbd. $(\alpha - \varepsilon, \alpha + \varepsilon)$ of α, contains $a_n = \alpha$, for an infinite number of values of n. Therefore, α is a limit point of $\langle a_n \rangle$.

Case II. Let S be an infinite set.

The range S being an infinite bounded set has a limit point, say p, So each nbd $(p - \varepsilon, p + \varepsilon)$ of p contains an infinite number of elements of S

i.e., $a_n \in (p - \varepsilon, p + \varepsilon)$ for an infinite number of values of n.

Hence p is a limit point of $\langle a_n \rangle$.

Remark

An unbounded sequence may or may not have a limit point.

Counter example, Since $a_n = n$ is an unbounded sequence with no limit point and $a_n = 1$, if n is even; $a_n = n$, if n is odd is an unbounded sequence with a limit point 1.

7. LIMIT OF A SEQUENCE

Definition

A sequence $< a_n >$ is said to have a limit 'l' if for sufficiently large values of n, $| a_n - l |$ can be made as small as we please.

l is the limit of a sequence, if for given $\varepsilon > 0 \; \exists \; n_0 \in N$ s.t.

$$| a_n - l | < \varepsilon, \qquad \forall \; n > n_0$$

or $\quad \lim a_n = l$

Some Important Limits

1. $\lim\limits_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e$

2. $\lim\limits_{n \to \infty} n^{1/n} = 1,$

3. $\lim\limits_{n \to \infty} \frac{1}{n^p} = 0$, when $p > 0$ and $p \in \mathbf{R}$

4. $\lim\limits_{n \to \infty} r^n = 0$, when $| r | < 1$ and $r \in \mathbf{R}$

5. $\lim\limits_{n \to \infty} r^{1/n} = 1$, when $r > 0$ and $r \in \mathbf{R}$

Example : Let $a_n = \frac{1}{n}$, and let us show that $\lim a_n = 0$.

Solution : Given an $\varepsilon > 0$, let us choose a N such that $\frac{1}{N} < \varepsilon$.

Now, if $n \geq \mathbf{N}$, then we have

$$|a_n - 0| = \left|\frac{1}{n}\right| = \frac{1}{n} \leq \frac{1}{N} < \varepsilon,$$

which is exactly what we needed to show to conclude that $\lim a_n = 0$.

Example : Let $a_n = (2n + 1)/(1 - 3n)$, and let $L = -2/3$. Let us show that $L = \lim a_n$.

Solution : Indeed, if $\varepsilon > 0$ is given, we must find a **N**, such that if $n \geq \mathbf{N}$ then $|a_n + (2/3)| < \varepsilon$. Let us examine the quantity $|a_n + 2/3|$. Maybe we can make some estimates on it, in such a way that it becomes clear how to find the natural number **N**.

$$|a_n + (2/3)| = \left|\frac{2n+1}{1-3n} + \frac{2}{3}\right|$$

$$= \left|\frac{6n+3+2-6n}{3-9n}\right|$$

$$= \left|\frac{5}{3-9n}\right|$$

$$= \frac{5}{9n-3}$$

$$= \frac{5}{6n+3n-3}$$

$$\leq \frac{5}{6n}$$

$$\leq \frac{1}{n},$$

for all $n \geq 1$. Therefore, if **N** is an integer for which $\mathbf{N} > 1/\varepsilon$, then

$|a_n + 2/3| < 1/n \leq 1/\mathbf{N} < \varepsilon$,

whenever $n \geq \mathbf{N}$, as desired.

Example: Let $a_n = 1/\sqrt{n}$, and let us show that $\lim a_n = 0$.

Solution : Given an $\varepsilon > 0$, we must find an integer **N** that satisfies the requirements of the definition. It's a little trickier this time to choose this **N**. Consider the positive number ε^2. We know, that there exists a natural number **N** such that $1/\mathbf{N} < \varepsilon^2$. Now, if $n \geq \mathbf{N}$, then

$$|a_n - 0| = \frac{1}{\sqrt{n}} \leq \frac{1}{\sqrt{N}} = \sqrt{\frac{1}{N}} < \sqrt{\varepsilon^2} = \varepsilon,$$

which shows that $0 = \lim 1/\sqrt{n}$.

Theorem : Prove that a bounded sequence with a unique limit point is convergent.

Proof : Let l be the unique limit point of a bounded sequence $\langle a_n \rangle$. Then for any $\varepsilon > 0$.

$a_n \in (l - \varepsilon, l + \varepsilon)$ for infinitely many values of n.

We show that there exists only finitely many values of n, say $m_1, m_2, \ldots, m_k$ such that $a_{m_1}, a_{m_2}, \ldots, a_{m_k}$ do not belong to $(l - \varepsilon, l + \varepsilon)$ reason being the infinitely many terms of the given sequence not belonging to $(l - \varepsilon, l + \varepsilon)$ will have a limit point other then l (by **Bolzano-Weierstrass Theorem**), which is a contradiction.

Let $m - 1 = \max\{m_1, m_2, \ldots, m_k\}$. It follows that

$a_n \in (l - \varepsilon, l + \varepsilon)$ for all $n \geq m$

i.e. $|a_n - l| < \varepsilon$ for all $n \geq m$.

Hence $\langle a_n \rangle$ is convergent to l.

8. LIMIT SUPERIOR AND LIMIT INFERIOR OF SEQUENCE

Let $\{a_n\}$ be a bounded sequence then the sequence has the least and the greatest limit point. The least limit point of $\{a_n\}$ is called limit infinite of $\{a_n\}$ and is denoted by $\underset{n\to\infty}{\text{Lt}} \inf a_n$.

The greatest limit point $\{a_n\}$ is called limit superior of $\{a_n\}$ and is denoted by $\underset{n\to\infty}{\text{Lt}} \sup a_n$.

Note : 1. If $\{a_n\}$ is unbounded above the $\underset{n\to\infty}{\text{Lt}} \sup a_n = \infty$

If $\{a_n\}$ is unbounded below then $\underset{n\to\infty}{\text{Lt}} \inf a_n = -\infty$.

2. Since the the greatest limit point of sequence $\{a_n\} \geq$ the least limit point of sequence $\{a_n\}$

$\underset{n\to\infty}{\text{Lt}} \sup a_n \geq \underset{n\to\infty}{\text{Lt}} \inf a_n$.

Ex. (i) $a_n = (-1)^n$.

Set of limit point = $\{-1, 1\}$

$\therefore$ $\underset{n\to\infty}{\text{Lt}} \sup a_n = 1$ and $\underset{n\to\infty}{\text{Lt}} \inf a_n = -1$.

(ii) $a_n = \dfrac{1}{n}$

Set of limit point = $\{0\}$

$\therefore$ $\underset{n\to\infty}{\text{Lt}} \sup a_n = 0 = \underset{n\to\infty}{\text{Lt}} \inf a_n$.

(iii) $a_n = (-1)^n$

As sequence is neither bounded above nor bounded below.

$\underset{n\to\infty}{\text{Lt}} \sup a_n = +\infty$ and $\underset{n\to\infty}{\text{Lt}} \inf a_n = -\infty$

Note :

(i) If sequence is bounded below then $\underset{n\to\infty}{\text{Lt}} \inf a_n$ = infimum of set of limit point of sequence a_n.

(ii) If sequence is bounded above then $\underset{n\to\infty}{\text{Lt}} \sup a_n$ = supremum of set of limit point of sequence a_n.

(iii) If sequence is bounded then $\underset{n\to\infty}{\text{Lt}} \sup a_n$ = supremum of set of limit point of sequence a_n.

$\underset{n\to\infty}{\text{Lt}} \inf a_n$ = infimum of set of limit points of sequence a_n.

Classification of Sequence

1. A sequence is said to be convergent $\Leftrightarrow$

$$\underset{n\to\infty}{\text{Lt}} \sup a_n = \underset{n\to\infty}{\text{Lt}} \inf a_n \quad \text{(finitely)}$$

2. A sequence is said to be divergent $\Leftrightarrow$

$$\underset{n\to\infty}{\text{Lt}} \sup a_n = \underset{n\to\infty}{\text{Lt}} \inf a_n \quad \text{(infinitely)}$$

3. A sequence is said to be oscillatory $\Leftrightarrow$

$$\underset{n\to\infty}{\text{Lt}} \sup a_n \neq \underset{n\to\infty}{\text{Lt}} \inf a_n.$$

Important

A sequence of purely irrational number can converge to a rational number

And, a sequence of purely rational number can converge to a irrational number.

Let $\alpha \in \mathbb{Q}^c$

for any $n \in \mathbb{N}$

$$x_n \in \left(\alpha - \frac{1}{n}, \alpha + \frac{1}{n}\right) \cap \mathbb{Q} \setminus \alpha$$

then $|x_n - \alpha| < \frac{1}{n}$ ...(1)

Any By Archimedean property ; for any $\varepsilon > 0$

$\exists\ m \in \mathbb{N}$ such that

$$\frac{1}{n} < \varepsilon \ \forall\ n \geq m \qquad ...(2)$$

From (1) & (2);

$$|x_n - \alpha| < \frac{1}{n} < \varepsilon\ \forall\ n \geq m$$

i.e. $|x_n - \alpha| < \varepsilon\ \forall\ n \geq m$

$$\therefore \quad \underset{n\to\infty}{\text{Lt}}\ x_n = \alpha\ ;\ \begin{matrix} x_n \neq \alpha \\ x_n \in \mathbb{Q} \end{matrix} \text{ for } n \in \mathbb{N}$$

Hence, sequence of purely rational number can converge to irrational number.

Similarly, we can prove the other statement.

9. CONVERGENT AND DIVERGENT SEQUENCE

Convergent Sequence

Definition : A sequence of real numbers is said to **converge** to a real number L if for every $\varepsilon > 0$ there is an integer $N > 0$ such that if $k > N$ then $|a_k - L| < \varepsilon$. The number L is called the **limit** of the sequence.

If $\{a_k\}$ converges to L we will write $\lim\limits_{k \to \infty} a_k = L$ or simply $a_k \to L$. If a sequence does not converge, then we say that it **either diverge or oscillate.**

Note that the N in the definition depends on the ε that we were given. If you change the value of ε then you may have to "recalculate" N.

Consider the sequence $a_n = \frac{n}{2^n}$, $n \in \mathbb{N}$. Now, if we look at the values that the sequence takes

$\frac{1}{2}, \frac{2}{2^2}, \frac{3}{2^3}, \frac{4}{2^4}, \ldots$

n	$\approx n/2^n$
1	0.5
2	0.5
3	0.375
4	0.25
5	0.15625
6	0.09375
7	0.0546875
8	0.03125
9	0.0175781
10	0.00976562

we might think that the terms are getting smaller and smaller so maybe the limit of this sequence would be 0. Let's take a look and compare how N would vary as ε varies. Let's start with some simple small numbers and let ε be 0.1, 0.01, 0.001, and 0.0001, and 0.00001.

For $\varepsilon = 0.1$, we need to find an integer N so that

$$\left|\frac{N}{2^N} - 0\right| < 0.1$$

Look in the table of values here and we see that for N = 6 we have satisfied the above condition. Following this we get the following by using a calculator or a computer algebra system:

$N > 0$ implies $\left|\frac{N}{2^N} - 0\right| < 1$

$N > 5$ implies $\left|\frac{N}{2^N} - 0\right| < 0.1$

$N > 9$ implies $\left|\frac{N}{2^N} - 0\right| < 0.01$

$$N > 14 \text{ implies } \left|\frac{N}{2^N} - 0\right| < 0.001$$

$$N > 18 \text{ implies } \left|\frac{N}{2^N} - 0\right| < 0.0001$$

$$N > 22 \text{ implies } \left|\frac{N}{2^N} - 0\right| < 0.00001$$

A sequence is said to be convergent if its limit is finite.

$$\lim_{n\to\infty} x_n = l$$

Definition : A sequence $< a_n >$ converges to any number l iff for given $\varepsilon > 0$ $\exists n_0 \in \mathbf{N}$ (depending on ε)

$$|a_n - l| < \varepsilon \qquad \forall\ n_0 > n$$

Divergent Sequence

A sequence is said to be divergent if $\lim\limits_{n\to\infty} a_n = +\infty$ or $-\infty$

Definition : A sequence is said to be divergent if and only if for any $k > 0$ $\exists$ a no. n_0 s.t.

$a_n > k\ \forall\ n > n_0$.

For Example,

(i). The sequence $\{a_n\} = \{n\} = \{1, 2, 3, 4, 5, 6, 7, ...\}$ diverges since its limit is infinity (∞).

(ii). The sequence $\{a_n\} = \{n^2\} = \{1, 4, 9, 16, 25, 36, 49, ...\}$ diverges

(iii). $V_n = (-1)^n$:

This sequence diverges whereas the sequence is bounded :

$-1 \le V_n \le 1$

(iv). Another example would be $x_n = \ln n$, since

$\ln n \to \infty$ but $\ln n - \ln(n-1) = \ln\dfrac{n}{n-1} \to \ln 1 = 0.$

(v). The sequence $\{\sin(n\pi/2)\}_{n\ge1}$ diverges because the sequence is

$\{1, 0, -1, 0, 1, 0, ...\}$,

and hence it does not converge to any number.

Oscillatory Sequence

The sequence which is neither convergent nor divergent is said to be an oscillatory sequence. i.e., Sequences that tend to nowhere are always oscillating sequences.

For ex. (i). $(-1)^n \to$ nowhere (ii). $(-2)^n \to$ nowhere

We remark that an unbounded sequence that does not diverge to ∞ or $-\infty$ oscillates infinity. For example, the sequences

$\{(-1)^n n\}$, $\{(-1)^n n^2\}$, $\{(-n)^n\}$

are all unbounded and oscillate infinitely.

Theorem : Every convergent sequence is bounded.

Proof : Let us suppose that $\langle a_n \rangle$ be a convergent sequence and

$$\lim_{n\to\infty} a_n = l.$$

Let $\varepsilon = 1$. Then there exists a positive integer m such that

$|a_n - l| < 1 \quad \forall\, n \geq m$

$\Rightarrow \quad l - 1 < a_n < l + 1 \quad \forall\, n \geq m.$ ...(1)

Let $K = \max\{a_1, a_2, ..., a_{m-1}, l + 1\}$, ...(2)

$k = \min\{a_1, a_2, ..., a_{m-1}, l - 1\}$. ...(3)

From (1), (2), (3), we get

$k \leq a_n \leq K \quad \forall\, n.$

Hence $\langle a_m \rangle$ is bounded.

Converse of this theorem is not true, counter example $a_n = (-1)^n$

Theorem : If $\lim\limits_{n\to\infty} a_n = l$, then l is the unique limit point of $\langle a_n \rangle$

Proof : Since $\lim\limits_{n\to\infty} a_n = l$, there exists a positive integer i such that

$|a_n - l| < \varepsilon/2 \;\forall\, n \geq i.$ ...(1)

Since l' is a limit point of $\langle a_n \rangle$, there exists a positive integer j such that

$a_n \in (l' - \frac{1}{2}\varepsilon, l' + \frac{1}{2}\varepsilon)$ for $n > j$ (i.e., for infinitely many values of j).

$\Rightarrow \quad |a_n - l'| < \varepsilon/2$ for $n > j$. ...(2)

Let $N > \max\{ i, j \}$

Since $N > i$, so by (1), $|a_N - l| < \varepsilon/2$ ...(3)

And since $N > j$, so by (2), $|a_N - l'| < \varepsilon/2$ ...(4)

Now $|l - l'| = |(a_N - l') - (a_N - l)|$

$\leq |a_N - l'| + |a_N - l| < \varepsilon/2 + \varepsilon/2 = \varepsilon$, by (3) and (4).

Since ε is arbitrarily small and $|l - l'| < \varepsilon$, so $|l - l'| = 0$ i.e., $l = l'$.

Hence l is the unique limit point of $\langle a_n \rangle$.

Converse of this theorem also doesn't hold, such as $a_n = 1$, if n is even; $a_n = n$, if n is odd is a sequence with a unique limit point 1 but it does not converge to 1.

Theorem : A necessary and sufficient condition for the convergent of a monotonic sequence is that it is bounded.

Proof : The condition is necessary($\Rightarrow$)

Let $\langle a_n \rangle$ be a monotonic and convergent sequence.

We know that every convergent sequence is bounded.

Hence $\langle a_n \rangle$ is bounded.

The condition is sufficient($\Leftarrow$)

Let $\langle a_n \rangle$ be a monotonic and bounded sequence.

Since $\langle a_n \rangle$ is monotonic, we may suppose that $\langle a_n \rangle$ is monotonically increasing. We shall prove that $\langle a_n \rangle$ is convergent.

Let $S = \{a_n : n \in \mathbf{N}\}$ be the range of $\langle a_n \rangle$.

Then S is bounded above, since the sequence $\langle a_n \rangle$ is bounded above By order completeness property, S has the least upper bound.

Let p = l.u.b. S. We shall show that $\langle a_n \rangle$ converges to p. Let $\varepsilon > 0$ be any number. Since $p - \varepsilon < p$, $p - \varepsilon$ cannot be an upper bound of S and so there exists some $a_m \in S$ such that $a_m > p - \varepsilon$.

Since $\langle a_n \rangle$ is monotonically increasing, therefore

$a_n \geq a_m \quad \forall\, n \geq m$

or $\quad a_n \geq a_m > p - \varepsilon \quad \forall\, n \geq m$ or $a_n > p - \varepsilon \;\forall\, n \geq m$ ...(i)

Also $\quad p = \sup S \quad \Rightarrow a_n \leq p < p + \varepsilon \quad \forall\, n.$ ...(ii)

$\therefore \quad p - \varepsilon < a_n < p + \varepsilon \quad \forall\, n \geq m$, using (i), (ii)

or $\quad |a_n - p| < \varepsilon \quad \forall \quad n \geq m$

Hence $\langle a_n \rangle$ converges to p.

Theorem : (i) Every monotonically increasing sequence which is bounded above converge to its least upper bound.

(ii) Every monotonically decreasing sequence which is bounded below converge to its greatest lower bound.

Proof. (i) Let $\{a_n\}$ be monotonically increasing sequence which is bounded above.

Let u be the least upper bound of sequence $\{a_n\}$

Let $\varepsilon > 0$

Since $\quad u - \varepsilon < u$

$\therefore \quad u - \varepsilon$ is not an upper bound of $\{a_n\}$

$\Rightarrow \quad \exists \;\; m \in \mathbb{R}$ s. that $a_n > u - \varepsilon$.

Since a_n is monotonically increasing

$a_n \geq a_m > u - \varepsilon \qquad \forall\, n \geq m$

$\Rightarrow \quad a_n > u - \varepsilon \qquad \forall\, n \geq m$ ...(1)

Also, u is least upper bound of $\{a_n\}$

$a_n \leq u \qquad \forall\, n \in \mathbb{N}$

$\Rightarrow \quad a_n \leq u + \varepsilon \qquad \forall\, n \geq m$ ...(2)

From (1) & (2)

$u - \varepsilon < a_n < u + \varepsilon \;\forall\, n \geq m$

$\Rightarrow \quad |a_n - u| < \varepsilon \;\forall\, n \geq m$

$\therefore \quad a_n \to u.$

Similarly, we can prove (2).

Ex. Find the value of α for which the sequence $\langle a_n \rangle$ converges to 1 where $a_1 = \alpha$, $a_{n+1} = \frac{1}{2}(a_n^2 + a_n)$ for $n \geq 1$

Sol. $\langle a_n \rangle = \langle 1, 1, 1, \dots \rangle$, if $\alpha = 1$. Hence $a_n \to 1$, if $\alpha = 1$.

Ex. If the sequence $\langle a_n \rangle$, where $a_n = 1 + \frac{1}{3} + \frac{1}{3^2} + \dots + \frac{1}{3^{n-1}} + \dots + \frac{1}{3^{n-1}}$ converges then what is the value of $\lim\limits_{n \to \infty} a_n$?

Sol. We have $a_n = 1 + \frac{1}{3} + \frac{1}{3^2} + \dots + \frac{1}{3^{n-1}}$,

which is a geometric progression with common ratio $\frac{1}{3}$.

$$\therefore \quad a_n = \frac{\left(1 - \frac{1}{3^n}\right)}{\left(1 - \frac{1}{3}\right)} = \frac{3}{2}\left(1 - \frac{1}{3^n}\right) = \frac{3}{2} - \frac{1}{2}\left(\frac{1}{3}\right)^{n-1}$$

Now $\lim\limits_{n \to \infty} a_n = \frac{3}{2} - \frac{1}{2} \lim\limits_{n \to \infty} \left(\frac{1}{3}\right)^{n-1} = \frac{3}{2} - \frac{1}{2} \times 0 = \frac{3}{2}$. [$\because$ $\lim r^n = 0$, if $|r| < 1$].

Hence $\langle a_n \rangle$ is convergent and $\lim\limits_{n \to \infty} a_n = \frac{3}{2}$.

Ex. The sequence (x_n) is bounded and monotone where $x_1 = \sqrt{2}$ and $x_{n+1} = \sqrt{2x_n}$, then it converges to 2.

Sol. Observe that $x_2 > x_1$. Since $x_{n+1}^2 - x_n^2 = 2(x_n - x_{n-1})$, by induction (x_n) is increasing. It can be observed again by induction that $x_n \le 2$.

Given $x_1 = \sqrt{2}$,

$x_{n+1} = \sqrt{2x_n}$

Let $\lim\limits_{n \to \infty} x_n = \ell$

then $\lim\limits_{n \to \infty} x_{n+1} = \lim\limits_{n \to \infty} \sqrt{2x_n}$

$\Rightarrow$ $\ell = \sqrt{2\ell}$

$\Rightarrow$ $\ell^2 = 2\ell$

$\Rightarrow$ $\ell^2 - 2\ell = 0$

$\Rightarrow$ $\ell(\ell - 2) = 0$

$\Rightarrow$ $\ell = 0, 2$

Since $\ell > 0$

$\therefore$ $\ell = 2$

10. THE ALGEBRA OF CONVERGENT SEQUENCES

Theorem. If the sequence $\{a_n\}$ converges to L and $c \in \mathbf{R}$, then the sequence $\{ca_n\}$ converges to cL; i.e., $\lim_{n\to\infty} ca_n = c \lim_{n\to\infty} a_n$.

Proof. Let's assume that $c \neq 0$, since the result is trivial if $c = 0$.

Let $\varepsilon > 0$. Since $\{a_n\}$ converges to L, we know that there is an $n \in \mathbf{N}$ so that if $n > N$.

$$|a_n - L| < \frac{\varepsilon}{|c|}.$$

Thus, for $n > N$ we then have that

$$|ca_n - cL| = |c||a_n - L| < |c|\frac{\varepsilon}{|c|} = \varepsilon.$$

which is what we needed to prove.

Theorem. If the sequence $\{a_n\}$ converges to L and $\{b_n\}$ converges to M, then the sequence $\{a_n + b_n\}$ converges to L + M; i.e., $\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$.

Proof. Let $\varepsilon > 0$. We need to find an $N \in \mathbf{R}$ so that if $n > N$

$$|(a_n + b_n) - (L + M)| < \varepsilon.$$

Since $\{a_n\}$ and $\{b_n\}$ are convergent, for the given ε there are integers $N_1, N_2 \in N$ so that

If $n > N_1$ then $|a_n - L| < \frac{\varepsilon}{2}$ and if $n > N_2$ then $|b_n - M| < \frac{\varepsilon}{2}$

Thus, if $n > \max\{N_1, N_2\}$ then

$$|(a_n + b_n) - (L + M)| \leq |a_n - L| + |b_n - M| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Theorem. If the sequence $\{a_n\}$ converges to L and $\{b_n\}$ converges to M, then the sequence $\{a_n \cdot b_n\}$ converges to L · M; i.e., $\lim_{n\to\infty}(a_n \cdot b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$.

The trick with the inequalities here is to look at the inequality

$$\begin{aligned} |a_nb_n - LM| &= |a_nb_n - a_nM + a_nM - LM| \\ &\leq |a_nb_n - a_nM| + |a_nM - LM| \\ &= |a_n||b_n - M| + |M||a_n - L|. \end{aligned}$$

So for large values of n, $|b_n - M|$ and $|a_n - L|$ are small and $|M|$ is constant. Now, by Theorem shows that $|a_n|$ is bounded, so that we will be able to show that $|a_nb_n - LM|$ is small.

Proof. Let $\varepsilon > 0$. Then there is a constant $K > 0$ such that $|a_n| \leq K$ for all n. Since $\{b_n\}$ is convergent, for the given ε there is an integer $N_1 \in \mathbf{N}$ so that

If $n > N_1$ then $|b_n - M| < \frac{\varepsilon}{2K}$.

Also, since $\{a_n\}$ is convergent there is an integer $N_2 \in \mathbf{N}$ so that

If $n > N_2$ then $|a_n - L| < \frac{\varepsilon}{2(|M|+1)}$.

Thus if $N = \max\{N_1, N_2\}$ then if $n > N$

$$|a_nb_n - LM| \le |a_n||b_n - M| + |M|\,|a_n - L|$$

$$\le K\cdot\frac{\varepsilon}{2K} + |M|\frac{\varepsilon}{2(|M|+1)} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Lemma. (limits preserve inequalities)

If the sequence $\{a_n\}$ converges to L and $\{b_n\}$ converges to M and if $a_n < b_n$ for all $n \ge m$, then $L \le M$.

Proof. Consider the sequence $\{c_n\}$ defined by $c_n := b_n - a_n$.

The Sum Rule for Limits grants that $c_n \to M - L$ as $n \to \infty$.

Furthermore, the assumption that $a_n = y_n$ for all $n \in N$ means that $c_n \ge 0$ for all $n \in N$.

Applying the main result to the sequence $\{c_n\}$ leads to the conclusion that $M - L \ge 0$.

That is, $L \le M$.

OR

Let if possible L>M.

Since lim a_n = L, therefore for $\varepsilon = (L-M)/2$, there exists n_1 such that $|a_n - L| < (L-M)/2$ for all $n \ge n_1$

i.e.. $L - (L-M)/2 < a_n < L + (L-M)/2$ for all $n \ge n_1$

i.e. $(L+M)/2 < a_n < (3L-M)/2$ for all $n \ge n_1$

Similarly, for b_n, there exists n_2 such that $|b_n - M| < (L-M)/2$ for all $n \ge n_2$

i.e. $M - (L-M)/2 < b_n < M + (L-M)/2$ for all $n \ge n_2$

i.e. $(-L+3M)/2 < b_n < (L+M)/2$ for all $n \ge n_2$

Let $N = \max(n_1, n_2)$

Then, for all $n \ge N$, $b_n < (L+M)/2 < a_n$

Let $M = \max(N, m)$

For all $n \ge M$, $b_n < a_n$, a contradiction

Hence $L \le M$.

Lemma. If the sequence $\{a_n\}$ converges to L, if $a_n \ne 0$ for all $n \in \mathbf{N}$, and if $L \ne 0$, then $\text{glb}\{|a_n| \mid n \in \mathbf{N}\} > 0$.

Proof. Let $\varepsilon = \frac{1}{2}|L| > 0$. Since $\{a_n\}$ converges to L, there is an $N \in \mathbf{N}$ so that if $n > N$ then $|a_n - L| < |L|/2$.

Now if $n > N$ we must have that $|a_n| \ge |L|/2$. If not then the triangle inequality would imply

$$|L| = |L - a_n + a_n| \le |L - a_n| + |a_n| < \frac{|L|}{2} + \frac{|L|}{2} = |L|.$$

Now we set

$$m = \min\left\{\frac{|L|}{2}, |a_1|, |a_2|, \ldots, |a_N|\right\}.$$

Then clearly $m > 0$ and $|a_n| \ge m$ for all $n \in \mathbf{N}$.

Theorem. If the sequence $\{a_n\}$ converges to L, if $a_n \ne 0$ for all $n \in \mathbf{N}$, and if $L \ne 0$, then the sequence $\{1/a_n\}$ converges to 1/L; i.e., $\lim\limits_{n\to\infty} 1/a_n = 1/\lim\limits_{n\to\infty} a_n$.

Proof. Let $\varepsilon > 0$. By Lemma there is a n, $m > 0$ such that $|a_n| \ge m$ for all n. Since $\{a_n\}$ is convergent there is an integer $N \in \mathbf{N}$ so that if $n > N$ $|L - a_n| < \varepsilon\cdot m|L|$.

Then for $n > N$

$$\left|\frac{1}{a_n} - \frac{1}{L}\right| = \frac{|a_n - L|}{|a_n L|} \le \frac{|a_n - L|}{m|L|} < \varepsilon.$$

Theorem. Suppose that the sequence $\{b_n\}$ converges to M and if $\{a_n\}$ converges to L. If $b_n \neq 0$ for all $n \in \mathbf{N}$, and if $M \neq 0$, then the sequence $\{a_n/b_n\}$ converges to L/M; i.e., $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim\limits_{n\to\infty} a_n}{\lim\limits_{n\to\infty} b_n}$.

Proof. We use two of the previous theorem to prove this. By Theorem $\{1/b_n\}$ converges to 1/M, so

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} a_n \cdot \frac{1}{b_n} = L \cdot \frac{1}{M} = \frac{L}{M}.$$

Theorem : Let $\{a_n\}$ be a sequence converging to a then the sequence $\{|a_n|\}$ converges to $|a|$:

Proof : This one, we break into three cases.

If $a > 0$, then (applying the definition of $\lim_{n\to\infty} a_n = a$ with $\varepsilon = a$) there exists M_0 so that $a_n > 0$ for $n > M_0$. In this case, we have $|a_n| = a_n$ for $n > M_0$ and $|a| = a$, and so $||a_n| - |a|| = |a_n - a|$. Since there is M_1 so that $|a_n - a| < \varepsilon$ for $n > M_1$, we have that $||a_n| - |a|| < \varepsilon$ for $n > M = \max(M_0, M_1)$, and so $\lim_{n\to\infty} |a_n| = |a|$.

If $a < 0$, then (applying the definition of $\lim_{n\to\infty} a_n = a$ with $\varepsilon = |a|$) there exists M_0 so that $a_n < 0$ for $n > M_0$. In this case, we have $|a_n| = -a_n$ for $n > M_0$ and $|a| = -a$, and so $||a_n| - |a|| = |-a_n + a| = |a_n - a|$. Since there is M_1 so that $|a_n - a| < \varepsilon$ for $n > M_1$, we have that $||a_n| - |a|| < \varepsilon$ for $n > M = \max(M_0, M_1)$, and so $\lim_{n\to\infty} |a_n| = |a|$.

If $a = 0$, then the definition of $\lim_{n\to\infty} a_n = a$ becomes : for every $\varepsilon > 0$, there exists M so that $|a_n - 0| = |a_n| < \varepsilon$ for $n > M$. Since $||a_n|| = |a_n|$, we have that the definition of $\lim_{n\to\infty} = 0$ is satisfied without any further work.

OR

Let $\varepsilon > 0$ be given. Since $\lim a_n = a$, therefore there exists n_1 such that $|a_n - a| < \varepsilon$ for all $n \geq n_1$. Hence, $|\,|a_n| - |a|\,| < |a_n - a| < \varepsilon$ for all $n \geq n_1$. Thus, $\lim |a_n| = |a|$

Note : Converse of the theorem holds only for $a = 0$. Thus, $\lim a_n = 0$ iff $\lim |a_n| = 0$. Converse does not hold in general. For example, if $a_n = (-1)^n$, then $\lim |a_n| = 1$ but a_n does not even converge.

Example. Let $p > 0$ then $\lim\limits_{n\to\infty} \frac{1}{n^p} = 0$.

Let $\varepsilon > 0$ and let $N = \left(\frac{1}{\varepsilon}\right)^{1/p}$. Then $n > N$ implies that $n^p > \frac{1}{\varepsilon}$ and hence $\varepsilon > \frac{1}{n^p}$. Since $\frac{1}{n^p} > 0$, this shows that $n > N$ implies $\left|\frac{1}{n^p} - 0\right| < \varepsilon$.

Example. Let $|a| < 1$, then $\lim\limits_{n\to\infty} a^n = 0$.

Suppose $a \neq 0$, because $\lim a^n = 0$ is clear for $a = 0$. Since $|a| < 1$, we can write

$|a| = \frac{1}{1+|b|}$ where $b > 0$. By the binomial theorem

$$(1+b)^n = 1 + nb + \frac{n(n-1)}{2}b^2 + \ldots + b^n \geq 1 + nb > nb,$$

so $\quad |a^n - 0| = |a^n| = \dfrac{1}{(1+b)^n} < \dfrac{1}{nb}$.

Now consider $\varepsilon > 0$ and let $N = \dfrac{1}{\varepsilon b}$.

Then if $n > N$, we have $n > \dfrac{1}{\varepsilon b}$ and hence $|a^n - 0| < \dfrac{1}{nb} < \varepsilon$.

Example. $\lim\limits_{n\to\infty} \left(n^{1/n}\right) = 1$.

Let $a_n = (n^{1/n}) - 1$ and note that $a_n \geq 0$ for all n.

By Theorem it is sufficient for us to show that $\lim_{n\to\infty} a_n = 0$.

Since $1 + a_n = n^{1/n}$, we have $n = (1 + a_n)^n$.

For $n \geq 2$ the binomial theorem gives us

$$n = (1 + a_n)^n \geq 1 + na_n + \frac{1}{2}n(n-1)a_n^2 > \frac{1}{2}n(n-1)a_n^2.$$

Thus, $n > \dfrac{1}{2}n(n-1)a_n^2$, so $a_n^2 < \dfrac{2}{n-1}$.

Thus, we have shown that $a_n < \sqrt{\dfrac{2}{n+1}}$ for $n \geq 2$.

Thus, $\lim a_n = 0$.

Example. $\lim\limits_{n\to\infty} (a^{1/n}) = 1$ for $a > 0$.

Suppose $a \geq 1$ then for $n \geq a$ we have $1 \leq a^{1/n} \leq n^{1/n}$. Since $\lim n^{1/n} = 1$, it easily follows that $\lim a^{1/n} = 1$. Suppose that $0 < a < 1$. Then $\dfrac{1}{a} > 1$, so that $\lim\left(\dfrac{1}{a}\right)^{1/n} = 1$.

Thus, $\quad \lim\left(\dfrac{1}{a}\right)^{1/n} = 1$

$$\lim \frac{1^{1/n}}{a^{1/n}} = 1$$

$$\frac{1}{\lim a^{1/n}} = 1$$

$$\lim a^{1/n} = 1$$

Recall : For a sequence $\{a_n\}$ we write $\lim a_n = +\infty$ provided for each $M > 0$ there is a number N such that $n > N$ implies that $a_n > M$.

In this case we will say that $\{a_n\}$ **diverges** to $+\infty$.

We can make a similar definition for $\lim a_n = -\infty$.

Of course, we cannot use the previous theorem when dealing with infinite limits.

Theorem. Let $\{a_n\}$ and $\{b_n\}$ be sequence such that $\lim a_n = +\infty$ and $\lim b_n > 0$. Then $\lim a_nb_n = +\infty$.

Proof. Let $M > 0$. Choose a real number m so that $0 < m < \lim b_n$. Whether $\lim b_n = +\infty$ or not, there exists N_1 so that if $n > N_1$ then $b_n > m$. Since $\lim a_n = +\infty$ there is an N_2 so that if $n > N_2$ then

$$a_n > \frac{M}{m}.$$

Setting N = max$\{N_1, N_2\}$ means that for n > N, $a_n b_n > \frac{M}{m} \cdot m = M$.

Theorem. For a sequence $\{a_n\}$ of positive real numbers lim $a_n = +\infty$ if and only if lim $\frac{1}{a_n} = 0$.

Proof. Let $\{a_n\}$ be a sequence of positive numbers. We need to show

If lim $a_n = +\infty$ then lim $\frac{1}{a_n} = 0$ ...(1)

and If lim $\frac{1}{a_n} = 0$ then lim $a_n = +\infty$. ...(2)

To prove (1) we will suppose that lim $a_n = +\infty$. Let $\varepsilon > 0$ and let $M = 1/\varepsilon$. Since $\{a_n\}$ diverges to $+\infty$, there is an N so that if n > N then $a_n > M = 1/\varepsilon$. Therefore, if N > n then $\varepsilon - 1/a_n > 0$, so

$$\text{if } n > N \text{ then } \left|\frac{1}{a_n} - 0\right| < \varepsilon.$$

Thus, this proves lim $1/a_n = 0$.

To prove (2) suppose that lim $1/a_n = 0$ and let M > 0. Let $\varepsilon = 1/M$. Then since $\varepsilon > 0$ there is an N so that if n > N then $\left|\frac{1}{a_n} - 0\right| < \varepsilon = \frac{1}{M}$. Since $a_n > 0$ we then know that

$$\text{if } n > N \text{ then } 0 < \frac{1}{a_n} < \frac{1}{M}$$

and hence

if n > N then $a_n > M$.

This means that lim $a_n = +\infty$ and 2 holds.

Theorem. All bounded monotone sequences converge.

Proof. Let $\{a_n\}$ be a bounded monotonically increasing sequence and let $S = \{a_n ; n \in \mathbf{N}\}$. Since the sequence is bounded, $a_n < M$ for some real number M and for all $n \in \mathbf{N}$. This means that the set S is bounded, and thus it has a least upper bound.

Let u = lub S and $\varepsilon > 0$. Since u = lub S and $\varepsilon > 0$, $u - \varepsilon$ is not an upper bound for S. This means that there must be some N so that $a_N > u - \varepsilon$. Since $\{a_n\}$ is monotonically increasing we have that for all n > N $a_n \geq a_N$ and hence for all n > N it follows that $u - \varepsilon < a_n \leq u$. Thus, $|a_n - u| < \varepsilon$ for all n > N. Thus, lim a_n = u = lub S.

The proof for bounded monotonically decreasing sequences is the same with the greatest lower bound playing the role of the least upper bound.

Theorem. Let $\{a_n\}$ be a sequence of real numbers.

(i) If $\{a_n\}$ is an unbounded monotonically increasing sequence, then lim $a_n = +\infty$.

(ii) If $\{a_n\}$ is an unbounded monotonically decreasing sequence, then lim $a_n = -\infty$.

Let $\{a_n\}$ be a bounded sequence of real numbers. While it may converge or may not converge, the limiting behavior of $\{a_n\}$ depends only on the "tails" of the sequence, or sets of the form $\{a_n \mid n > N\}$. This leads us to a concept that if a given sequence converges or diverges.

Let $u_N = \text{glb}\{a_n \mid n > N\} = \inf\{a_n \mid n > N\}$ and let $v_N = \text{lub}\{a_n \mid n > N\} = \sup\{a_n \mid n > N\}$. We have seen that if $\lim a_n$ exists, then it must lie in the interval $[u_N, v_N]$. As N increases, the sets $\{a_n \mid n > N\}$ get smaller, so we have

$$u_1 \le u_2 \le u_3 \le \ldots \text{ and } v_1 \ge v_2 \ge v_3 \ge \ldots$$

By the above theorem the limits $u = \lim_{N\to\infty} u_N$ and $v = \lim_{N\to\infty} v_N$ both exist and $u \le v$ since $u_N \le v_N$ for all N. If the limit exists, then $u_N \le \lim a_n \le v_N$ exists or not.

Theorem. A monotonic sequence of real numbers $\{x_n\}$ converges if and only if it is bounded.

Proof : We already know that a convergent sequence is bounded. Hence we just need to show that a monotone and bounded sequence of real numbers is convergent.

Suppose $\{x_n\}$ is a bounded increasing sequence.

Let S denote the non-empty bounded set $\{x_n : n \in \mathbb{N}\}$.

By the completeness axiom S has a least upper bound, and we let $x = \sup S$.

We claim that $\lim x_n = x$. Given any $\varepsilon > 0$, $x - \varepsilon$ is not an upper bound for S.

Thus there exists an integer N such that $x_N > x - \varepsilon$.

Furthermore, since $\{x_n\}$ is increasing and x is an upper bound for S we have

$$x - \varepsilon < x_N \le x_n \le x < x + \varepsilon$$

or equivalently

$$|x_n - x| < \varepsilon$$

for all $n \ge N$. Hence $\lim x_n = n$.

In the case when the sequence is decreasing, let $x = \inf S$ and proceed in a similar manner.

Lemma (On nested Intervals). Let the sequence of the intervals $\{[a_n, b_n]\}$ be such that $[a_{n+1}, b_{n+1}] \subset [a_n, b_n]$ $\forall n \in \mathbf{N}$ (such intervals are called nested), and their lengths converge to zero, i.e., $\lim_{n\to\infty}(b_n - a_n) = 0$. Then there exists a unique point ξ that belongs to every interval, i.e., $\xi \in [a_n, b_n]$ $\forall n \in \mathbf{N}$, and

$$\xi = \lim_{n\to\infty} a_n = \sup\{a_n\} = \lim_{n\to\infty} b_n = \inf\{b_n\}.$$

Proof. The sequence $\{a_n\}$ and $\{b_n\}$ are monotonic and bounded:

$$a_1 \le a_n \le a_{n+1} \le b_{n+1} \le b_n \le b_1 \quad \forall n \in \mathbf{N}.$$

By Theorem, they are convergent, and

$$\lim_{n\to\infty} a_n = \sup\{a_n\} = a, \quad \lim_{n\to\infty} b_n = \inf\{b_n\} = b.$$

However, $\lim_{n\to\infty}(b_n - a_n) = 0$ implies that $a = b =: \xi$, and clearly

$$a_n \le a = \xi = b \le b_n \quad \forall n \in \mathbf{N}.$$

Next we show that there is no point other than ξ that belongs to all the intervals $[a_n, b_n]$. Assume the opposite: $\exists \eta \in [a_n, b_n]$ $\forall n$ and $\eta \ne \xi$. Then by Theorem

$$b_n - a_n \ge |\xi - \eta| > 0 \ \ \forall n \ \Rightarrow \ \lim_{n\to\infty}(b_n - a_n \ge |\xi - \eta| > 0.$$

This contradiction completes the proof of the lemma.

Example : Determine if the following sequences are monotonic and/or bounded.

(a) $\left\{-n^2\right\}_{n=0}^{\infty}$

(b) $\left\{(-1)^{n+1}\right\}_{n=1}^{\infty}$

(c) $\left\{\dfrac{2}{n^2}\right\}_{n=5}^{\infty}$

Solution.

(a) $\{-n^2\}_{n=0}^{\infty}$

This sequence is a decreasing sequence (and hence monotonic) because,

$$-n^2 > -(n+1)^2$$

for every n.

Also, since the sequence terms will be either zero or negative this sequence is bounded above. We can use any positive number or zero as the bound, M, however, it's standard to choose the smallest possible bound if we can and it's a nice number. So, we'll choose M = 0 since,

$$-n^2 \le 0 \qquad \text{for every n}$$

This sequence is not bounded below however since we can always get below any potential bound by taking n large enough. Therefore, while the sequence is bounded above it is not bounded below.

we can also note that this sequence diverges (to $-\infty$).

(b) $\{(-1)^{n+1}\}_{n=1}^{\infty}$

The sequence terms in this sequence alternate between 1 and -1 and so the sequence is neither an increasing sequence or a decreasing sequence. Since the sequence is neither an increasing nor decreasing sequence it is not a monotonic sequence. It is an oscillatory sequence.

The sequence is bounded however since it is bounded above by 1 and bounded below by –1.

Again, we can note that this sequence is not divergent, it is oscillatory.

(c) $\left\{\dfrac{2}{n^2}\right\}_{n=5}^{\infty}$

This sequence is a decreasing sequence (and hence monotonic) since,

$$\frac{2}{n^2} > \frac{2}{(n+1)^2}$$

The terms in this sequence are all positive and so it is bounded below by zero. Also, since the sequence is a decreasing sequence the first sequence term will be the largest and so we can see that the sequence will also be bounded above by 2/25. Therefore, this sequence is bounded.

We can also take a quick limit and note that this sequence converges and its limit is zero.

Example. Determine if the following sequences are monotonic and/or bounded.

(a) $\left\{\dfrac{n}{n+1}\right\}_{n=1}^{\infty}$

(b) $\left\{\dfrac{n^3}{n^4+10000}\right\}_{n=0}^{\infty}$

Solution.

(a) $\left\{\dfrac{n}{n+1}\right\}_{n=1}^{\infty}$

We'll start with the bounded part of this example first and then come back and deal with the increasing/decreasing.

First, n is positive and so the sequence terms are all positive. The sequence is therefore bounded below by zero. Likewise each sequence term is the quotient of a number divided by a larger number and so is guaranteed to be less than one. The sequence is then bounded above by one. So, this sequence is bounded.

Now let's think about the monotonic question. First, we will often make the mistake of assuming that because the denominator is larger the quotient must be decreasing. This will not always be the case and in this case we would be wrong. This sequence is increasing.

To determine the increasing/decreasing nature of this sequence we will need to calculate

I technique. First consider the following function and its derivative.

$$f(x) = \frac{x}{x+1} \qquad f'(x) = \frac{1}{(x+1)^2}$$

We can see that the first derivative is always positive and so from calculation we know that the function must then be an increasing function. Notice that,

$$f(n) = \frac{n}{n+1} = a_n$$

Therefore because n < n + 1 and f(n) is increasing we can also say that,

$$a_n = \frac{n}{n+1} = f(n) < f(n+1) = \frac{n+1}{n+1+1} = a_{n+1} \quad \Rightarrow \quad a_n < a_{n+1}$$

In other words, the sequence must be increasing.

Note that now that we know the sequence is an increasing sequence we can get a better lower bound for the sequence. Since the sequence is increasing the first term in the sequence must be the smallest term and so since we are starting at n = 1 we could also use a lower bound of ½ for this sequence. It is important to remember that any number that is always less than or equal to all the sequence terms can be a lower bound.

(b) $\left\{\frac{n^3}{n^4+10000}\right\}_{n=0}^{\infty}$

This is a messy looking

Now, let's move on to the increasing/decreasing part. As with the problem, many of us will look at the exponents in the numerator and denominator and determine based on that sequence terms must decrease.

This however, isn't a decreasing sequence. Let's take a look at the first few terms to see this.

$$a_1 = \frac{1}{10001} \approx 0.00009999 \qquad a_2 = \frac{1}{1252} \approx 0.0007987$$

$$a_3 = \frac{27}{10081} \approx 0.005678 \qquad a_4 = \frac{4}{641} \approx 0.006240$$

$$a_5 = \frac{1}{85} \approx 0.011756 \qquad a_6 = \frac{27}{1412} \approx 0.019122$$

$$a_7 = \frac{343}{12401} \approx 0.02766 \qquad a_8 = \frac{32}{881} \approx 0.03632$$

$$a_9 = \frac{729}{16561} \approx 0.04402 \qquad a_{10} = \frac{1}{20} \approx 0.05$$

The first 10 terms of this sequence are all increasing and so clearly the sequence can't be a decreasing sequence. Recall that a sequence can only be decreasing if ALL the terms are decreasing.

Now, we can't make another common mistake and assume that because the first few terms increase then whole sequence must also increase. If we did that we would also be mistaken as this is also not an increasing sequence.

This sequence is neither decreasing or increasing. The only sure way to see this is to do the Calculus approach to increasing/decreasing functions.

In this case we'll need to the following function and its derivative.

$$f(x) = \frac{x^3}{x^4 + 10000} \qquad f'(x) = \frac{-x^2\left(x^4 - 30000\right)}{\left(x^4 + 10000\right)^2}$$

This function will have the following three **critical points,**

$$x = 0,\ x = \sqrt[4]{30000} \approx 13.1607,\ \ x = -\sqrt[4]{30000} \approx -13.1607$$

Remember these are the only points where the function *may* change sign! Our sequence starts at n = 0 and so we can ignore the third one since it lies outside the values of n that we're considering. By plugging in some test values of x we can quickly determine that the derivative is positive for $0 < x < \sqrt[4]{30000} \approx 13.16$ and so the function is increasing in this range. Likewise, we can see that the derivative is negative for $x > \sqrt[4]{30000} \approx 13.16$ and so the function will be decreasing in this range.

So, our sequence will be increasing for $0 \le n \le 13$ and decreasing for $n \ge 13$. Therefore the function is not monotonic.

Finally, note that this sequence will also converge and has a limit of zero.

Example : Let $x_1 = 6$ and $x_{n+1} = 5 - \frac{6}{x_n}$ for all $n \in \mathbf{N}$ then the sequence converges to.

Solution : $x_1 = 6$

$$x_2 = 5 - \frac{6}{x_1} = 5 - \frac{6}{6} = 5 - 1 = 4$$

$$x_3 = 5 - \frac{6}{x_2} = 5 - \frac{6}{4} = 5 - \frac{3}{2} = \frac{7}{2} = 3.5$$

$$x_4 = 5 - \frac{6}{x_3} = 5 - \frac{6 \times 2}{7} = \frac{35 - 12}{7} = \frac{23}{7} = 3.28$$

$$x_5 = 5 - \frac{6}{x_4} = 5 - \frac{6 \times 7}{23} = \frac{115 - 42}{23} = \frac{73}{23} = 3.17$$

Let $x_k > 3$ for some $k \in \mathbb{N}$

then $x_{k+1} > 5 - \frac{6}{x_k} = 5 - \frac{6}{3} = 5 - 2 = 3$

We have $x_1 > 3$ and if we assume that $x_k > 3$ for some $k \in \mathbf{N}$, then $x_{k+1} > 5 - 2 = 3$. Hence by the principle of mathematical induction, $x_n > 3$ for all $n \in \mathbf{N}$. Therefore (x_n) is bounded below. Again, x_2

$= 4 < x_1$ and if we assume that $x_{k+1} < x_k$ for some $k \in \mathbf{N}$, then $x_{k+2} - x_{k+1} = 6\left(\frac{1}{x_k} - \frac{1}{x_{k+1}}\right) < 0 \Rightarrow x_{k+2} < x_{k+1}$. Hence by the principle of mathematical induction, $x_{k+1} < x_n$ for all $n \in \mathbf{N}$. Therefore (x_n) is decreasing. $<x_n>$ is bounded and monotonic. Hence $<x_n>$ is convergent. Let $\ell = \lim\limits_{n\to\infty} x_n$. Then $\lim\limits_{n\to\infty} x_{n+1} = 5 - \frac{6}{\lim\limits_{n\to\infty} x_n} \Rightarrow$ $\ell = 5 - \frac{6}{\ell}$ (since $x_n > 3$ for all $n \in \mathbf{N}$, $\ell \neq 0$) $\Rightarrow (\ell - 2)(\ell - 3) = 0 \Rightarrow \ell = 2$ or $\ell = 3$. But $x_n > 3$ for all $n \in \mathbf{N}$, so $\ell \geq 3$. Therefore $\ell = 3$.

Example : The series $\sum u_k = \left(\frac{2}{5^{k+1}} + \frac{(2k)!}{3^k}\right)$ converges or diverges

Solution : The given series is a sum of two series. We will look at each of them separately and hope that the two results can be put together. First consider the series

$$\sum \frac{2}{5^{k+1}}$$

This is a series that can be converted to a geometric series.

$$\sum \frac{2}{5^{k+1}} = \frac{2}{5}\sum \frac{1}{5^k} = \frac{2}{5}\sum\left(\frac{1}{5}\right)^k$$

Since q = 1/5 satisfies |q| < 1, this series converges.

Now we will look at the series

$$\sum \frac{(2k)!}{3^k}$$

When we see factorial, we immediately think of the Ratio test.

$$\lambda = \lim_{k\to\infty}\left(\frac{a_{k+1}}{a_k}\right) = \lim_{k\to\infty}\left(\frac{\frac{(2(k+1))!}{3^{k+1}}}{\frac{(2k)!}{3^k}}\right) = \lim_{k\to\infty}\left(\frac{(2k+2)!}{(2k)!}\frac{3k}{3^{k-1}}\right)$$

$$= \lim_{k\to\infty}\left(\frac{1\cdot 2\cdot 3\cdot \ldots \cdot (2k)\cdot (2k+1)\cdot (2k+2)}{1\cdot 2\cdot 3\cdot \ldots \cdot (2k)}\frac{1}{3}\right)$$

$$= \lim_{k\to\infty}\left(\frac{(2k+1)(2k+2)}{3}\right) = \infty.$$

Since $\lambda > 1$, this series diverges.

Since the given series is the sum of a convergent and a divergent series, it is divergent.

11. SUBSEQUENCE

Definition : Let $\langle a_n \rangle$ be any sequence. Let $\langle n_1, n_2, ..., n_k ...\rangle$ be a sequence of positive integers such that $i > j \Rightarrow n_i > n_j$. Then the sequence $\langle a_{n_1}, a_{n_2}, ..., a_{n_k} ....\rangle$, written as $\langle a_{n_k} \rangle$, is called a **subsequence** of $\langle a_n \rangle$.

Let $\{a_i\}_{i=1}^{\infty}$ be a sequence. When we extract from this sequence only certain elements and drop the remaining ones we obtain a new sequences consisting of an infinite subset of the original sequence.

That sequence is called a **subsequence** and denoted by $\{a_{i_k}\}_{k=1}^{\infty}$

$\langle a_{10}, a_{11}, a_{12}, ...\rangle$ is a subsequence of $\langle a_n \rangle$.

Theorem : If sequence bdd., then every subsequence is bdd and conversely.

Proof. Let $\{a_n\}$ be any bdd. sequence

then $\exists$ m, M $\in \mathbb{R}$ such that

$$m \le a_n \le M \ \forall\ n \in \mathbb{N} \quad ...(1)$$

Let $\{a_{n_k}\}$ be any subsequence of $\{a_n\}$

$\Rightarrow \quad m \le a_{n_k} \le m$ (from (1)) $\forall\ n_k \in \mathbb{N}$

$\therefore \quad \{a_{n_k}\}$ is also bdd subsequence

Conversely

Let every subsequence of $\{a_n\}$ be bdd.

As every sequence is a subsequence of itself.

$\therefore \quad \{a_n\}$ is bdd.

Note : If we have to show $\{a_n\}$ is unbounded.

Then show $\exists$ a subsequence of $\{a_n\}$ which is unbounded.

Ex. $\quad a_n = 1 + \frac{1}{2} + ... + \frac{1}{n}$

Consider

$$a_{2^2} = 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{2^n}$$

$$= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + ... + \left(\frac{1}{2^{n-1}+1} + ... + \frac{1}{2^n}\right)$$

$$> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + ... + \frac{1}{8}\right) + ... + \left(\frac{1}{2^n} + ... + \frac{1}{2^n}\right)$$

$$= 1 + \frac{1}{2} + 2 \cdot \frac{1}{2^2} + 2^2 \cdot \frac{1}{2^3} + ... + 2^n \frac{1}{2^n}$$

$$= 1 + \frac{n}{2}$$

$$\therefore \quad a_{2^n} > 1 + \frac{n}{2}\ \forall\ n > 2$$

So, $\{a_{2^n}\}$ is unbounded

Hence, $\{a_n\}$ is unbounded.

Observation

(i). A sequence has a constant subsequence iff $\exists$ a real no. which appears in the sequence infinite many times.

(ii). If range of a sequence is a finite set then there exist a real no. which appears in the sequence infinite many times.

Theorem : If a sequence $\langle a_n \rangle$ converges to l, then every subsequence of $\langle a_n \rangle$ converges to l.

Proof : Let $\langle a_n \rangle$ be a subsequence of $\langle a_n \rangle$.

Since $a_n \to l$, therefore, for any $\varepsilon > 0$, there exists a positive integer m such that

$$|a_n - l| < \varepsilon \qquad \forall\ n \geq m.$$

In particular, $|a_{n_k} - l| < \varepsilon \qquad \forall\ n_k \geq m.$

Hence $\langle a_{n_k} \rangle$ converges to l.

Remark : The converse of the above theorem may not be true.

Theorem : Every bounded sequence has a convergent subsequence.

Proof : Let $<x_n>$ is bounded

Case 1 : When the range of the sequence is finite.

If the range of $<x_n>$ in finite then at least one element 'a' repeats infinitely because a sequence has infinite number of terms, if we choose this element 'a' again and again then the subsequence $<a, a, \ldots>$ formed is convergent.

Case 2 : When the range of the sequence is infinite

$<x_n>$ is bounded and range is infinite bounded subset of set of real no. **R** we know that set of bounded real no. has at least one limit point. Therefore, let l is the limit point of $<x_n>$.

For each $n \in \mathbf{N}$, $\exists\ v_n \in \mathbf{N}$ s.t.

Let $<x_{v_1}, x_{v_2}, \ldots x_{v_n}, \ldots>$ be a subsequence of $<x_n>$.

$\therefore$ for given $\varepsilon > 0\ \exists\ n_0 \in \mathbf{N}$ s.t. $\dfrac{1}{n_0} < \varepsilon$

then $|x_{v_n} - l| < \dfrac{1}{n} < \dfrac{1}{n_0} < \varepsilon,\ \forall\ n > n_0.$ s.t. $\dfrac{1}{n_0} < \varepsilon$

$\lim\limits_{x \to \infty} x_{v_n} = l$

$\Rightarrow$ $<x_{v_1}, x_{v_2}, \ldots x_{v_n}, \ldots>$ is convergent.

Theorem : If the subsequences $\{a_{2n-1}\}$ and $\{a_{2n}\}$ of a sequence $\{a_n\}$ converge to the same limit l, then the sequence $\{a_n\}$ converges to l.

Proof. Let $\varepsilon > 0$ be given

$\{a_{2n-1}\}$ converges to l

$\Rightarrow$ For $\varepsilon > 0$, $\exists$ a positive integer m_1 such that $|a_{2n-1} - l| < \varepsilon\ \forall\ n \geq m_1$

$\{a_{2n-1}\}$ converges to l

$\Rightarrow$ For $\varepsilon > 0$, $\exists$ a positive integer m_2 such that $|a_{2n} - l| < \varepsilon\ \forall\ n \geq m_2$

Let $m = \max.\{m_1, m_2\}$, then $|a_{2n-1} - l| < \varepsilon$ and $|a_{2n} - l| < \varepsilon\ \forall\ n \geq m$

$\Rightarrow$ $|a_n - l| < \varepsilon \quad \forall\ n \geq m$

$\Rightarrow$ $\{a_n\}$ converges to l.

Ex. Take the sequence $\{(-1)^j\}_{j=1}^{\infty}$. Extract every other member, starting with the first. Then do the same, starting with the second.

The sequence in question is

$$\{(-1)^j\}_{j=1}^{\infty} = \{-1, 1, -1, 1, -1, 1, ...\}$$

If we extract every second number, starting with the first, we get:

$\{-1, -1, -1, -1, ...\}$

This subsequence now converges to -1.

If we extract every second number, starting with the second, we get:

$\{1, 1, 1, 1, 1, ...\}$

This subsequence now converges to 1.

- **Examples :** The sequence $\left(\left(\frac{1}{j^2}, \frac{1}{j^2}\right)\right)_{j=1}^{\infty}$ in R^2 is a subsequence of the sequence $\left(\left(\frac{1}{n}, \frac{1}{n}\right)\right)_{j=1}^{\infty}$ (in this case, $n_j := j^2$). The sequence 1, 1, 1, 1, ... is a subsequence of 1, 0, 1, 0, 1, ...

Ex. Take the sequence $\left\{\frac{1}{j}\right\}_{j=1}^{\infty}$. Extract three different subsequences of your choice and look at the convergence behavior of these subsequences.

The sequence in question is :

$$\left\{\frac{1}{j}\right\}_{j=1}^{\infty} = \{1, 1/2, 1/3, 1/4, 1/5, 1/6, ...\}$$

which converges to zero. Now let us extract some subsequences:

Extracting the even terms yields the subsequence

$\{1/2, 1/4, 1/6, 1/8, 1/10, ...\}$

which converges to zero.

Extracting the odd terms yields the subsequence

$\{1, 1/3, 1/5, 1/7, 1/9, ...\}$

which converges to zero.

Extracting every third member yields the sequence

$\{1, 1/4, 1/7, 1/10, 1/13, ...\}$

which converges to zero.

Hence, all three subsequences converge to zero. This is an illustration of a general result: If a sequence converges to a limit L then every subsequence extracted from it will also converge to that limit L.

- On the other hand, it is possible for a subsequence to be convergent without the sequence as a whole being convergent. For example, the sequence 1, 0, 1, 0, 1, ... is not convergent, even though certain subsequences of it (such as 1, 1, 1, ... converges).

12. CAUCHY SEQUENCE

Definition. A sequence $<a_n>$ of real numbers is said to be a Cauchy sequence if for every $\varepsilon >$ 0 there exists a natural number N such that for all natural numbers n, m > N, the terms a_n, a_m satisfy $|a_n - a_m| < \varepsilon$.

Definition. A sequence $\langle a_n \rangle$ is said to be a **Cauchy sequence** if for any $\varepsilon > 0$, there exists a positive integer m such that

$$|a_n - a_m| < \varepsilon, \text{ whenever } n \geq m.$$

Note: It can be easily shown that the two definitions are equivalent. Most of the authors use the 1st one as definition but in practice we'll often use the latter one.

Theorem : Cauchy Sequence is bounded.

Proof : Let $\langle a_n \rangle$ be a Cauchy sequence.

For $\varepsilon = 1$, there exists a positive integer m such that

$$|a_n - a_m| < 1, \qquad \text{whenever } n \geq m,$$

or $\quad a_m - 1 < a_n < a_m + 1, \qquad$ whenever $n \geq m$.

Let $\quad k = \min \{a_1, a_2, ..., a_{m-1}, a_m - 1\}$,

$\quad K = \max \{a_1, a_2, ..., a_{m-1}, a_m + 1\}$.

Then $\quad k \leq a_n \leq K$ for all n.

Hence $\langle a_n \rangle$ is bounded.

Theorem : If a subsequence of a Cauchy sequence converges to l, then the sequence also converges to l.

Proof : Let $< x_n >$ be a cauchy sequence, then by definition for any $\varepsilon > 0 \; \exists \; n_o$ s.t.

$$|x_m - x_n| < \varepsilon/2 \qquad \forall m,n > 0 \qquad ...(1)$$

Let $< x_{v_n} >$ be the subsequence of $< x_n >$ and $\lim x_{v_n} = l$, then for $l > 0 \; \exists \; n_o \in \mathbf{N}$ s.t.

$$|x_n - l| < \varepsilon/2 \qquad \forall v_n > n_0 \qquad ...(2)$$

Now $\quad |x_n - l| = |x_n l| \leq |x_{v_n} - x_n| + |x_{v_n} - l|$

$\quad < \varepsilon/2 + \varepsilon/2 \qquad \forall n > v_n \quad$ [by (1) and (2)]

$\Rightarrow \quad |x_n - l| < \varepsilon \qquad \forall n > v_n$

$\Rightarrow \quad \lim x_n = l$

Theorem : A sequence $X = <x_n>$ in **R** is convergent if and only if it's Cauchy.

Proof : "$\Rightarrow$"

If $x := \lim X$, then given $\in > 0$ there is a natural number K such that if $n \geq K$ then $|Xn - x| < \in/2$. Thus, if $n, m \geq K$, then we have $|x_n - x_m| = |(x_n - x) + (x - x_m)| \leq |x_n - x| + |x_m - x| < \in/2 + \in/2 = \in$. Since $\in > 0$ is arbitrary, it follows that $<x_n>$ is a Cauchy sequence.

(Here we used the first definition)

"$\Leftarrow$" let $X = (x_n)$ be a Cauchy sequence; we will show that X is convergent to some real number. First we observe that the sequence X is bounded. Therefore, by the Bolzano-Weierstrass Theorem, there is a subsequence $X' = < x_{n_k} >$ of X that converges to some real number x*. We shall complete the proof by showing that X converges to x*.

Since $X = (x_n)$ is a Cauchy sequence, given > 0 there is a natural number N such that if n, m N then $|x_n - x_m| <$.

Since the subsequence X' = () converges to x*, there is a natural number K>N belonging to the set $\{n_1, n_2, \ldots\}$ such that $|x_K - x^*| <$.

Thus nN, $|x_n - x_K| < |x_n - x^*| |x_n - x_K| + |x_K - x^*| <$.

Hence, X converges to x*.

Theorem : (Cauchy's Principle of Convergence)

A necessary and sufficient condition for a series $\sum u_n$ to converge is that to each $\varepsilon > 0$, there exists a positive integer m, such that $|S_n - S_m| \le \varepsilon$

$$|u_{m+1} + u_{m+2} + \ldots + u_n| < \varepsilon, \text{ for all } n \ge m.$$

Proof : Let $\langle S_n \rangle$ be the sequence of partial sums of the series $\sum u_n$. By definition, $\sum u_n$ converges $\Leftrightarrow \langle S_n \rangle$ converges

$\Leftrightarrow$ To each $\varepsilon > 0$, there exists a positive integer m such that

$|S_n - S_m| < \varepsilon \ \forall\ n \ge m$

(Cauchy's Principle of Convergence for Sequences)

$\Leftrightarrow$ $|(u_1 + u_2 + \ldots + u_m + u_{m+1} + u_{m+2} + \ldots + u_n) - (u_1 + u_2 + \ldots + u_m)| < \varepsilon \ \forall\ n \ge m$

$\Leftrightarrow$ $|u_{m+1} + u_{m+2} + \ldots + u_n| < \varepsilon \ \forall\ n \ge m.$

Hence the theorem.

Examples

1. The sequence $\left\langle 1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots \right\rangle$ is a Cauchy sequence.

2. The sequence $\langle n^2 \rangle$ is not a Cauchy sequence.

3. The sequence $\langle a_n \rangle = \langle (-1)^n \rangle = \langle -1, 1, -1, 1, \ldots \rangle$ is not a Cauchy sequence, for if $\langle a_n \rangle$ is a Cauchy sequence, then for $\varepsilon = 1$, there exists a positive integer m such that

$$|a_n - a_m| < 1 \ \forall\ n \ge m.$$

If m is an even integer, then $a_m = 1$.

We can choose $n = 2m > m$ for which $a_n = 1$ and

$$|a_n - a_m| = |1 + 1| = 2 \le 1, \text{ a contradiction.}$$

Hence $\langle (-1)^n \rangle$ is not a Cauchy sequence.

4. Consider the series (that is, infinite sum)

$$S = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

We may view this as the limit of the sequence of partial sums

$$a_j = \sum_{n=1}^{j} \frac{1}{n^2}.$$

We can show that the limit converges using Theorem by showing that $\{a_j\}$ is a Cauchy sequence. Observe that if j, k > N, we definitely have

$$|a_j - a_k| \le \sum_{n=N}^{\infty} \frac{1}{n^2}.$$

It may be difficult to get an exact expression for the sum on the right, but it is easy to get an upper bound.

$$\sum_{n=N}^{\infty} \frac{1}{n^2} \le \sum_{n=N}^{\infty} \frac{1}{n(n-1)} = \sum_{n=N}^{\infty} \frac{1}{n-1} - \frac{1}{n}.$$

The reason we used the slightly wasteful inequality, replacing $\frac{1}{n^2}$ by $\frac{1}{n^2-n}$ is that now the sum on the right telescopes, and we know it is exactly equal to $\frac{1}{N-1}$. To sum up, we have shown that when j, k > N, we have

$$|a_j - a_k| \le \frac{1}{N-1}.$$

5. Let $0 < \alpha < 1$ and let (x_n) satisfy the condition $|x_{n+1} - x_n| \le \alpha^n$ for all $n \in \mathbf{N}$ then (x_n) is cauchy.

Sol. For all $m, n \in \mathbf{N}$ with $m > n$, we have $|x_m - x_n| \le |x_n - x_{n+1}| + |x_{n+1} - x_{n+2}| + \ldots + |x_{m-1} - x_m| \le \alpha^n + \alpha^{n+1} + \ldots + \alpha^{m-1} = \frac{\alpha^n}{1-\alpha}(1 - \alpha^{m-n}) < \frac{\alpha^n}{1-\alpha}$. Since $0 < \alpha < 1$, $\alpha^{m-n} \to 0$ and so given any $\varepsilon > 0$, we can choose $n_0 \in \mathbf{N}$ such that $\frac{\alpha^n}{1-\alpha}(1 - 0) < \varepsilon$. Hence for all m, n $\ge n_0$, we have $|x_m - x_n| < \frac{\alpha^n}{1-\alpha}(1 - 0) < \varepsilon$. Therefore (x_n) is a Cauchy sequence.

6. Let $x_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots.. + \frac{1}{n!}$ for all $n \in \mathbf{N}$ then (x_n) is ______

Sol. For all $m, n \in \mathbf{N}$ with $m > n$, we have $|x_m - x_n| = \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \ldots + \frac{1}{m!} \le \frac{1}{2^n} + \frac{1}{2^{n+1}} + \ldots + \frac{1}{2^{m-1}} = \frac{2}{2^n}\left(1 - \frac{1}{2^{m-n}}\right) < \frac{2}{2^n} < \frac{2}{n}$. Given $\varepsilon > 0$, we choose $n_0 \in \mathbf{N}$ satisfying $n_0 > \frac{2}{\varepsilon}$. Then for all $m, n \ge n_0$, we get $|x_m - x_n| < \frac{2}{n_0} < \varepsilon$. Consequently (x_n) is a Cauchy sequence in **R** and hence (x_n) is convergent.

7. Show that the sequence $\langle a_n \rangle$ defined by $a_n = (-1)^n/n$ converges.

Sol. If we show that $\langle a_n \rangle$ is a Cauchy sequence, then by Cauchy's General Principle of Convergence, $\langle a_n \rangle$ converges.

Let $\varepsilon > 0$. If $n \ge m$, then $1/n \le 1/m$ and

$$|a_n - a_m| = \left|\frac{(-1)^n}{n} - \frac{(-1)^m}{m}\right| \le \left|\frac{(-1)^n}{n}\right| + \left|\frac{(-1)^m}{m}\right|$$

$$\le \frac{1}{n} + \frac{1}{m} \le \frac{1}{m} + \frac{1}{m} = \frac{2}{m} < \varepsilon, \text{ if } m > \frac{2}{\varepsilon}.$$

Let m be a positive integer $> 2/\varepsilon$. Then

$$|a_n - a_m| < \varepsilon \ \forall \ n \ge m.$$

So $\langle a_n \rangle$ is a Cauchy sequence and hence convergent.

Example : (a) Show that if $\{a_n\}_{n=1}^{\infty}$ is Cauchy then $\{a_n^2\}_{n=1}^{\infty}$ is also Cauchy.

(b) Given an example of a Cauchy sequence $\{a_n^2\}_{n=1}^{\infty}$ such that $\{a_n\}_{n=1}^{\infty}$ is not Cauchy.

Solution : (a) Since $\{a_n\}_{n=1}^{\infty}$ is Cauchy, it is convergent. Since the product of two convergent sequences is convergent the sequence $\{a_n^2\}_{n=1}^{\infty}$ is convergent and therefore is Cauchy.

(b) Let $a_n = (-1)^n$ for all $n \in \mathbf{N}$. The sequence $\{a_n\}_{n=1}^{\infty}$ is not Cauchy since it is divergent. However, the sequence $\{a_n^2\}_{n=1}^{\infty} = \{1, 1, ...\}$ converges to 1 so it is Cauchy.

Example : Let $\{a_n\}_{n=1}^{\infty}$ be a Cauchy sequence such that a_n is an integer for all $n \in \mathbf{N}$. Show that there is a positive integer **N** such that $a_n = C$ for all $n \geq N$, where C is a constant.

Solution. Let $\varepsilon = \frac{1}{2}$. Since $\{a_n\}_{n=1}^{\infty}$ is Cauchy, there is a positive integer **N** such that if m, n > N we have $|a_m - a_n| < \frac{1}{2}$. But $a_m - a_n$ is an integer so we must have $a_n = a_N$ for all $n \geq \mathbf{N}$.

Example : What does it mean for a sequence $\{a_n\}_{n=1}^{\infty}$ to not be Cauchy?

Solution. A sequence $\{a_n\}_{n=1}^{\infty}$ is not a Cauchy sequence if there is a real number $\varepsilon > 0$ such that for all positive integers N there exist $n, m \in \mathbf{N}$ such that $n, m \geq N$ and $|a_n - a_m| < \varepsilon$.

Example : Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be two Cauchy sequences. Define $c_n = |a_n - b_n|$. Show that $\{c_n\}_{n=1}^{\infty}$ is a Cauchy sequence.

Solution. Let $\varepsilon > 0$ be given. There exist positive integers N_1 and N_2 such that if $n, m \geq N_1$ and $n, m \geq N_2$ we have $|a_n - a_m| < \frac{\varepsilon}{2}$ and $|b_n - b_m| < \frac{\varepsilon}{2}$. Let $N = N_1 + N_2$. If $n, m \geq N$ then $|c_n - c_m| = ||a_n - b_n| - |a_m - b_m|| \leq |(a_n - b_n) + (a_m - b_m)| \leq |a_n - a_m| + |b_n - b_m| < \varepsilon$. Hence $\{c_n\}_{n=1}^{\infty}$ is a Cauchy sequence.

Example : Explain why the sequence defined by $a_n = (-1)^n$ is not a Cauchy sequence.

Solution. We know that every Cauchy sequence is convergent. We also know that the given sequence is divergent. Thus, it can not be Cauchy.

Example : Every subsequence of a Cauchy sequence is itself a Cauchy sequence.

For a sequence $(s_n)_{n\in\mathbb{N}}$, a subsequence is a sequence of the form $(s_{n_k})_{k\in\mathbb{N}}$ for a strictly increasing sequence of natural numbers $1 \leq n_1 < n_2 < n_3 < \ldots$. For every integer $k \in \mathbb{N}$, by induction on k, n_k is at least as large as k.

First, let $(s_n)_{n\in\mathbb{N}}$ be a sequence that converges to s. Let $(s_{n_k})_{k\in\mathbb{N}}$ be a subsequence. For every $\varepsilon > 0$, since $(s_n)_{n\in\mathbb{N}}$ converges, there exists $N \in \mathbb{N}$ such that for every $n \in \mathbb{N}$ with $n \geq N$, $|s_n - s|$ is less than ε. For every $k \in \mathbb{N}$ with $k \geq N$, since n_k is at least as large as k, in particular $n_k = N$. Therefore, $|s_{n_k} - s|$ is less than ε. Thus $(s_{n_k})_{k\in\mathbb{N}}$ converges to s.

Next, let $(s_n)_{n\in\mathbb{N}}$ be a Cauchy sequence. For every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for every $m, n \in \mathbb{N}$ with $m \geq N$ and with $n \geq N$, $|s_n - s_m|$ is less than ε. For every $k, l \in \mathbb{N}$ with $k \geq N$ and with $l \geq N$, then $n_k \geq k \geq N$ and $n_l \geq l \geq N$. Therefore, $|s_{n_k} - s_{n_l}|$ is less than ε. Thus $(s_{n_k})_{k\in\mathbb{N}}$ is a Cauchy sequence.

Example : If a subsequence of a Cauchy sequence converges to L, then the full sequence also converges to L.

Example : Prove directly from the definition that the sequence

$$a_n = \frac{n+3}{2n+1},\ n \in \mathbb{N}$$

is a Cauchy sequence.

Solution : Let $\varepsilon > 0$ be given. Let N be a positive integer to be chosen. Suppose that $n, m \geq N$. We have

$$|a_n - a_m| = \left|\frac{n+3}{2n+1} - \frac{m+3}{2m+1}\right| = 3\frac{|m-n|}{(2n+1)(2m+1)}$$

$$\leq \frac{2m+2n}{(2n+1)(2m+1)} = \frac{(2n+1)+(2m+1)-2}{(2n+1)(2m+1)}$$

$$= \frac{1}{2m+1} + \frac{1}{2n+1} - \frac{2}{(2n+1)(2m+1)}$$

$$\leq \frac{1}{2m+1} + \frac{1}{2n+1}$$

$$\leq \frac{2}{2N+1}$$

Choose N so that $\frac{2}{2N+1} < \varepsilon$. This is $N > \frac{2-\varepsilon}{2\varepsilon}$. In this case,

$$|a_n - a_m| < \varepsilon$$

for all $n, m \geq N$. That is, $\left\{\frac{n+3}{2n+1}\right\}_{n=1}^{\infty}$ is Cauchy.

Example : Consider a sequence defined recursively by $a_1 = 1$ and $a_{n+1} = a_n + (-1)^n n^3$ for all $n \in \mathbb{N}$. Show that such a sequence is not a Cauchy sequence. Does this sequence converge?

Solution. We will show that there is an $\varepsilon > 0$ such that for all $N \in \mathbb{N}$ there exist m and n such that $m, n \geq N$ but $|a_m - a_n| \geq \varepsilon$. Note that $|a_{n+1} - a_n| = n^3 \geq 1$. Let $N \in \mathbb{N}$. Choose $m = N + 1$ and $n = N$. In this case, $|a_m - a_n| = N^3 \geq 1 = \varepsilon$. Hence, the given sequence is not a Cauchy sequence. Since every convergent sequence must be Cauchy, the given sequence is divergent.

Theorem : (Cauchy's first theorem on limits)

If $\lim\limits_{n\to\infty} a_n = l$, then $\lim\limits_{n\to\infty} \frac{a_1 + a_2 + \ldots + a_n}{n} = l$.

Proof : Let $b_n = a_n - l$ or $a_n = l + b_n\ \forall\ n \in \mathbf{N}$. ...(i)

Then $\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} a_n - l = l - l = 0$

$\therefore\quad \lim\limits_{n\to\infty} b_n = 0.$ ...(ii)

Now $\frac{a_1 + a_2 + \ldots + a_n}{n} = l + \frac{b_1 + b_2 + \ldots + b_n}{n}$, using (i)

To show that $\lim\limits_{n\to\infty}\left(\frac{a_1 + a_2 + \ldots + a_n}{n}\right) = l$, we shall prove that

$$\lim_{n\to\infty}\left(\frac{b_1+b_2+\ldots+b_n}{n}\right)=0.$$

Let $\varepsilon > 0$ be any number. Now $\lim\limits_{n\to\infty} b = 0$ implies that there exists a positive integer m such that

$$|b_n - 0| < \frac{\varepsilon}{2} \quad \forall \quad n \geq m \quad \Rightarrow \quad |b_n| < \frac{\varepsilon}{2} \quad \forall \quad n \geq m. \qquad \text{...(iii)}$$

Again $\lim\limits_{n\to\infty} b_n = 0 \Rightarrow \langle b_n\rangle$ is convergent $\Rightarrow \langle b_n\rangle$ is bounded

$\Rightarrow$ there exists a number $K > 0$ such that $|b_n| \leq K \ \forall \ n \in \mathbf{N}$. ...(iv)

Now $\left|\frac{b_1+b_2+\ldots+b_n}{n}\right| \leq \frac{1}{2}\left(|b_1|+|b_2|+\ldots+|b_m|\right) \leq \frac{mK}{n}$,

$\therefore \quad \left|\frac{b_1+b_2+\ldots+b_n}{n}\right| \leq \frac{\varepsilon}{2}$, when $\frac{mK}{n} < \frac{\varepsilon}{2}$, i.e., $n > \frac{2mK}{\varepsilon}$

Let p be a positive integer $> \frac{2mK}{\varepsilon}$. Then

$$\left|\frac{b_1+b_2+\ldots+b_m}{n}\right| < \frac{\varepsilon}{2} \quad \forall \quad n \geq p. \qquad \text{...(v)}$$

Now $\left|\frac{b_{m+1}+b_{m+2}+\ldots+b_n}{n}\right| \leq \frac{1}{n}(|b_{m+1}|+|b_{m+2}|+\ldots+|b_n|)$

$\leq \frac{1}{n}\left(\frac{\varepsilon}{2}+\frac{\varepsilon}{2}+\ldots+\frac{\varepsilon}{2}\right)$, by (iii)

(n – m) times

$= \left(\frac{n-m}{n}\right) \cdot \frac{\varepsilon}{2} < \frac{\varepsilon}{2} \ \forall \ n \geq m.$

$$\therefore \quad \left|\frac{b_{m+1}+b_{m+2}+\ldots+b_n}{n}\right| < \frac{\varepsilon}{2} \quad \forall \quad n \geq m. \qquad \text{...(vi)}$$

Let $\quad q = \max(m, p)$. Then ...(vii)

$$\left|\frac{b_1+b_2+\ldots+b_n}{n}\right| = \left|\frac{b_1+b_2+\ldots+b_m}{n} + \left|\frac{b_{m+1}+\ldots+b_n}{n}\right|\right|$$

$\leq \left|\frac{b_1+b_2+\ldots+b_m}{n}\right| = \left|\frac{b_{m+1}+\ldots+b_n}{n}\right| < \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon \ \forall \ n \geq q$, using (v), (vi), (vii)

$$\therefore \quad \left|\frac{b_1+b_2+\ldots+b_n}{n}\right| = \left|\frac{b_1+b_2+\ldots+b_n}{n} - 0\right| < \varepsilon \ \forall \ n \geq q.$$

Hence $\lim\limits_{n\to\infty}\left(\frac{b_1+b_2+\ldots+b_n}{n}\right) = 0$, which was to be proved.

Remark

The converse of the above theorem may not be true.

For example, $\langle a_n \rangle = \langle -1, 1, -1, 1, ... \rangle$ is not convergent

and $a_1 + a_2 = 0$, $a_1 + a_2 + a_3 = -1$, $a_1 + a_2 + a_3 + a_4 = 0$, etc.

Now $\frac{a_1 + a_2 + ... + a_n}{n} = 0$, if n is even

$= -\frac{1}{n}$, if n is odd.

Hence $\lim\limits_{n\to\infty}\left(\frac{a_1 + a_2 + ... + a_n}{n}\right) = 0$, but $\lim\limits_{n\to\infty} a_n \neq 0$.

Theorem : (Cauchy's second theorem on limits)

If $\lim\limits_{n\to\infty} a_n = l$, where $a_n > 0\ \forall\ n \in \mathbf{N}$, then $\lim\limits_{n\to\infty} (a_1, a_2, ... a_n)^{1/n} = l$.

Proof : $\lim\limits_{n\to\infty} a_n = l \Rightarrow \lim\limits_{n\to\infty} (\log a_n) = \log\left(\lim\limits_{n\to\infty} a_n\right) = \log l$.

By Cauchy's first theorem on limits, $\lim\limits_{n\to\infty} (\log a_n) = \log l$

$$\Rightarrow \lim_{n\to\infty}\left(\frac{\log a_1 + \log a_2 + ... + \log a_n}{n}\right) = \log l$$

$$\Rightarrow \lim_{n\to\infty}\left(\frac{1}{n}\log(a_1\ a_2\ ....\ a_n)\right) = \log l$$

$$\Rightarrow \lim_{n\to\infty}\left[\log(a_1 a_2\ ...\ a_n)^{\frac{1}{n}}\right] = \log l$$

$$\Rightarrow \log\left[\lim_{n\to\infty}(a_1 a_2\ ...\ a_n)^{\frac{1}{n}}\right] = \log l$$

Hence $\lim\limits_{n\to\infty}(a_1 a_2\ ...\ a_n)^{\frac{1}{n}} = l$.

Corollary : Suppose $\langle a_n \rangle$ is a sequence of positive real numbers such that

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = a(a > 0). \text{ Then } \lim_{n\to\infty}(a_n)^{1/n} = a.$$

Proof : Let $a_0 = 1$. Then $\lim\limits_{n\to\infty}\frac{a_n}{a_{n-1}} = a$.

By Theorem it follows that

$$\lim_{n\to\infty}\left[\frac{a_1}{a_0}\cdot\frac{a_2}{a_1}\cdot\frac{a_3}{a_2}\cdots\frac{a_n}{a_{n-1}}\right]^{\frac{1}{n}} = a.$$

Hence $\lim\limits_{n\to\infty}(a_n)^{1/n} = a$.

Example : Prove that $\lim\limits_{n\to\infty}\left\{\frac{(3n)!}{(n!)^3}\right\}^{1/n} = 27$.

Solution. We have $a_n = \dfrac{(3n)!}{(n!)^3}$, $a_{n+1} = \dfrac{(3n+3)!}{((n+1)!)^3}$

$$\therefore \qquad \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(3n+3)!}{((n+1)!)^3} \cdot \frac{(n!)^3}{(3n)!}$$

$$= \lim_{n\to\infty} \frac{(3n+3)(3n+2)(3n+1)}{(n+1)^3}$$

$$= \lim_{n\to\infty} \frac{(3+3/n)(3+2/n)(3+1/n)}{(1+1/n)^3} = 27.$$

Using the Corollary of Theorem we get

$$\lim_{n\to\infty} (a_n)^{1/n} = 27.$$

Example : Prove that if

$$a_n = \frac{1}{n}\{(n+1)(n+2)\ \ldots\ (n+1)\}^{1/n},$$

then $\langle a_n \rangle$ converges to $4/e$.

Solution. Let $b_n = \dfrac{(n+1)(n+2)\ldots(n+n)}{n^n}$, so that $(b_n)^{1/n} = a_n$.

We have $b_{n+1} = \dfrac{(n+2)(n+3)\ \ldots\ 2n(2n+1)(2n+2)}{(n+1)^{n+1}}$

$$\therefore \qquad \lim_{n\to\infty} \frac{b_{n+1}}{b_n} = \lim_{n\to\infty} \frac{n^n(n+2)(n+3)\ \ldots\ 2n(2n+1)(2n+2)}{(n+1)^{n+1}(n+1)\ (n+2)\ \ldots\ 2n}$$

$$= \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n \frac{(2n+1)(2n+2)}{(n+1)^2}$$

$$= \lim_{n\to\infty} \frac{1}{(1+1/n)^n} \frac{(2+1/n)(2+2/n)}{(1+1/n)^2} = \frac{4}{e}.$$

Hence $\lim_{n\to\infty} (b_n)^{1/n} = \lim_{n\to\infty} a_n = 4/e$.

Some important theorems on limits

Theorem : If $a_n \geq 0\ \forall\ n \in \mathbf{N}$ and $\lim_{n\to\infty} a_n = l$, then $l \geq 0$.

Proof : Let, if possible, $l < 0$.

Let $\quad \varepsilon = -l > 0$.

Since $\lim_{n\to\infty} a_n = l$, so there exists a positive integer m such that

$$|a_n - l| < -l \quad \forall \quad n \geq m$$

$\Rightarrow \quad a_n - l \leq |a_n - l| < -l \ \Rightarrow\ a_n < -l + l = 0 \quad \forall \quad n \geq m$

$\Rightarrow \quad a_n < 0 \quad \forall \quad n \geq m,$

which contradicts the fact that $a_n \geq 0\ \forall\ n$. Hence $l \geq 0$

Theorem : If $\langle a_n \rangle$ and $\langle b_n \rangle$ be two sequences such that $a_n \leq b_n\ \forall\ n \in \mathbf{N}$, then $\lim a_n \leq \lim b_n$

Proof : We see that $a_n \leq b_n \Rightarrow b_n \geq a_n \Rightarrow b_n - a_n \geq 0$

$\Rightarrow \quad \lim_{n\to\infty} (b_n - a_n) \geq 0,$

$\Rightarrow \quad \lim\limits_{n\to\infty} b_n - \lim\limits_{n\to\infty} a_n \geq 0 \Rightarrow \lim\limits_{n\to\infty} b_n \geq \lim\limits_{n\to\infty} a_n.$

Hence $\lim\limits_{n\to\infty} a_n \leq \lim\limits_{n\to\infty} b_n$.

Theorem : (Sandwich Theorem)

If $\langle a_n \rangle$, $\langle b_n \rangle$ and $\langle c_n \rangle$ be three sequences such that

(i) $a_n \leq b_n \leq c_n \ \forall \ n \in \mathbf{N}$, (ii) $\lim\limits_{n\to\infty} a_n = l = \lim\limits_{n\to\infty} c_n$

then $\lim\limits_{n\to\infty} b_n = l$.

Proof : Let $\varepsilon > 0$ be given.

Since $\lim\limits_{n\to\infty} b_n = l$, there exists a positive integer m_1 such that

$$|a_n - l| < \varepsilon \quad \forall \quad v \geq \mu_1$$

$$\Rightarrow \quad l - \varepsilon < a_n < l + \varepsilon \quad \forall \quad n \geq m_1.$$

Since $\lim\limits_{n\to\infty} c_n = l$, there exists a positive integer m_2 such that

$$|c_n - l| < \varepsilon \quad \forall \quad n \geq m_2$$

$$\Rightarrow \quad l - \varepsilon < c_n < l + \varepsilon \quad \forall \quad n \geq m_2$$

Let $m = \max\{m_1, m_2\}$. Then

$$l - \varepsilon < a_n < l + \varepsilon \quad \text{and} \quad l - \varepsilon < c_n < l + \varepsilon, \forall \ n \geq m. \qquad ...(*)$$

From (*) and condition (i), we obtain

$$l - \varepsilon < a_n \leq b_n \leq c_n < l + \varepsilon \quad \forall \quad n \geq m$$

$$\Rightarrow \quad l - \varepsilon < b_n < l + \varepsilon \quad \forall \quad n \geq m \Rightarrow |b_n - l| < \varepsilon \ \ \forall \ n \geq m.$$

Hence $\lim\limits_{n\to\infty} b_n = l$.

Theorem : Ratio test for convergence of sequence

If $\langle a_n \rangle$ be a sequence of positive number such that

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = l,$$

1. If $l < 1$, then $\lim\limits_{n\to\infty} (a_n) = 0$

2. If $l > 1$, then $\lim\limits_{n\to\infty} (a_n) = \infty$

3. If $l = 1$, then Test fails.

Proof : (1) Since $|l| < 1$, we can choose a positive number ε so small such that

$$|l| + \varepsilon + 1 \quad \text{or} \quad k < 1, \text{ where } k = |l| + \varepsilon. \qquad ...(1)$$

Since $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = l$, there exists a positive integer m such that

$$\left|\frac{a_{n+1}}{a_n} - l\right| < \varepsilon \ \ \forall \ \ n \geq m$$

$$\Rightarrow \quad \left|\frac{a_{n+1}}{a_n}\right| - |l| \leq \left|\frac{a_{n+1}}{a_n} - l\right| < \varepsilon \qquad (\because \ |x - y| \geq |x| - |y|)$$

$$\Rightarrow \quad \left|\frac{a_{n-1}}{a_n}\right| < |l| + \varepsilon = k, \text{ by (1).}$$

Putting n = m, m + 1, ... , (n – 1) successively and multiplying, we get

$$\left|\frac{a_{m+1}}{a_m}\right| \left|\frac{a_{m+2}}{a_{m+1}}\right| \ldots \left|\frac{a_n}{a_{n-1}}\right| < k \cdot k \ldots k = k^{n-m}$$

$$\Rightarrow \quad \left|\frac{a_n}{a_m}\right| < k^{n-m}$$

$$\Rightarrow \quad |a_n| < \frac{|a_m|}{k^m} \cdot k^n. \qquad \text{...(2)}$$

We know $k^n \to 0$ as $n \to \infty$, since $0 < k < 1$.

Using in (2), we see that $|a_n| \to 0$ as $n \to \infty$. Hence $\lim a_n = 0$.

Proof : (2) Since $l > 1$, we can choose a positive number ε such that

$l - \varepsilon > 1$ or $k > 1$, where $k = l - \varepsilon$.

Since $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = l$, there exists a positive integer m such that

$$\left|\frac{a_{n+1}}{a_n} - l\right| < \varepsilon, \quad \forall \ n \geq m$$

$$\Rightarrow \quad l - \varepsilon < \frac{a_{n+1}}{a_n} < l + \varepsilon, \quad \forall \ n \geq m.$$

Consider $\frac{a_{n+1}}{a_n} > l - \varepsilon = k \quad \forall \ n \geq m$.

$$\therefore \quad \frac{a_{n+1}}{a_n} > k \quad \forall \ n \geq m.$$

Putting n = m, m + 1, ... , (n – 1) successively and multiplying, we get

$$\left|\frac{a_n}{a_m}\right| > k^{n-m},$$

$$\Rightarrow \quad |a_n| > \frac{|a_m|}{k^m} k^n.$$

Now $k > 1 \Rightarrow k^n \to \infty$ as $n \to \infty$,

Hence $\lim\limits_{n\to\infty} a_n = \infty$.

Proof: (3). Let $a_n = 1 \ \forall \ n$

$< 1, 1, 1, \ldots > \to 1$ (constant sequence)

(this is a convergent sequence)

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \text{Lim}\frac{1}{1} = 1 = L$$

Consider $a_n = n \ \forall \ n$

$< 1, 2, 3, 4, \ldots n, n + 1, \ldots > \to \infty$

(this is a divergent sequence)

$$\frac{a_{n+1}}{a_n} = \frac{n+1}{n} = 1 + \frac{1}{n} \to 1 = L \text{ as } n \to \infty$$

So in this case we cannot conclude anything about the sequence.

Example : Deduce that $\lim\limits_{n\to\infty} 2^{-n} n^2 = 0$.

Solution : Let $a_n = \frac{n^2}{2^n}$ so that $a_{n+1} = \frac{(n+1)^2}{2^{n+1}}$

$$\therefore \qquad \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)^2}{2^{n+1}} \cdot \frac{2^n}{n^2} = \lim_{n\to\infty} \frac{1}{2}\left(\frac{n+1}{n}\right)^2$$

$$= \lim_{n\to\infty} \frac{1}{2}\left(1+\frac{1}{n}\right)^2 = \frac{1}{2} < 1.$$

Hence, by Theorem $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} 2^{-n} n^2 = 0$.

Example : Show that $\lim\limits_{n\to\infty} (3^{-n} n^3) = 0$.

Solution : Let $a_n = \frac{n^3}{3^n}$ so that $a_{n+1} = \frac{(n+1)^3}{3^{n+1}}$.

$$\therefore \qquad \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)^3}{3^{n+1}} \cdot \frac{3^n}{n^3} = \lim_{n\to\infty} \frac{1}{3}\left(1+\frac{1}{n}\right)^3 = \frac{1}{3} < 1.$$

Hence $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} 3^{-n} n^3 = 0$.

Example : Show that for any number x, $\lim\limits_{n\to\infty} \frac{x^n}{n!} = 0$.

Solution. Let $a_n = \frac{x^n}{n!}$, so that $a_{n+1} = \frac{x^{n+1}}{(n+1)!}$.

$$\Rightarrow \qquad \frac{a_{n+1}}{a_n} = \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} = \frac{x}{n+1}.$$

$$\therefore \qquad \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{x}{x+1} = 0 < 1.$$

Hence $\lim\limits_{n\to\infty} a_n = 0$ i.e., $\lim\limits_{n\to\infty} \frac{x^n}{n+1} = 0$.

Example : Show that

$$\lim_{n\to\infty} \frac{m(m-1)(m-2)\ldots(m-n+1)}{n!} x^n = 0, \text{ if } |x| < 1.$$

Solution : Let $a_n = \dfrac{m(m-1)\dots(m-n+1)}{n!}x^n$, so that

$$a_{n+1} = \frac{m(m-1)\dots(m-n+1)(m-n)}{(n+1)!}x^{n+1}.$$

Now $$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\left\{\frac{(m-n)}{(n+1)!}.n!.x\right\}$$

$$= \lim_{n\to\infty}\left(\frac{m-n}{n+1}\right)x = \lim_{n\to\infty}\frac{\left(\frac{m}{n}\right)-1}{1+\left(\frac{1}{n}\right)}.\ x = -x.$$

$\therefore$ $\displaystyle\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = -x$, where $|-x| = |x| < 1$.

Hence $\displaystyle\lim_{n\to\infty} a_n = 0$,

Example : Prove that if $p > 0$, then $\displaystyle\lim_{n\to\infty}\frac{n^k}{(1+p)^n} = 0$, k being a fixed real number.

Solution. Let $a_n = \dfrac{n^k}{(1+p)^n}$, so that $a_{n+1} = \dfrac{(n+1)^k}{(1+p)^{n+1}}$.

$$\therefore \quad \frac{a_{n+1}}{a_n} = \left(\frac{n+1}{n}\right)^k.\frac{1}{1+p} = \left(1+\frac{1}{n}\right)^k.\frac{1}{1+p}$$

Now $$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \frac{1}{(1+p)}.\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^k = \frac{1}{1+p},$$

where $\dfrac{1}{1+p} < 1$ $\quad (\because\ 1 + p > 1$ as $p > 0)$

Hence $\displaystyle\lim_{n\to\infty} a_n = 0$

Example : Show that $\displaystyle\lim_{n\to\infty}\frac{(1+y)^n}{n!} = 0$, for all y.

Solution. Let $a_n = \dfrac{(1+y)^n}{n!}$, so that $a_{n+1} = \dfrac{(1+y)^{n+1}}{(n+1)!}$

Now $$\frac{a_{n+1}}{a_n} = (1+y)\frac{n!}{(n+1)!} = \frac{1+y}{n+1}.$$

$$\therefore \quad \lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{1+y}{n+1} = 0 < 1.$$

Hence $\displaystyle\lim_{n\to\infty} a_n = 0$ i.e., $\displaystyle\lim_{n\to\infty}\frac{(1+y)^n}{n!} = 0$.

Unit-2 Series of Real Numbers

1. INTRODUCTION (SERIES OF REAL NUMBERS)

A finite series is given by the terms of a finite sequence, added together. For example, we could take the finite sequence

$$(2k+1)_{k=1}^{10} = (3, 5, 7 ... 21)$$

Then the corresponding example of a finite series would be given by all of these terms added together,

$$3 + 5 + 7 + ... + 21$$

We can write this sum more concisely using sigma notation. We write the capital Greek letter sigma, and then the rule for the kth term. Below the sigma we write 'k = 1'. Above the sigma we write the value of k for the last term in the sum, which in this case is 10. So in this case we would have

$$\sum_{k=1}^{10} 2k + 1 = 3 + 5 + 7 + ... + 21$$

and in this case the sum of the series is equal to 120

In the easy way, an infinite series is the sum of the terms of an infinite sequence. An example of an infinite sequence is

$$\left(\frac{1}{2^k}\right)_{k=1}^{\infty} = \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, ...\right),$$

And then the series obtained from this sequence would be

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + ...$$

With a sum going on forever. Once again we can use sigma notation to express this series. We write down the sigma sign and the rule for the k-th term. But now we put the symbol for infinity above the sigma, to show that we are adding up an infinite number of terms. In this case we would have

$$\sum_{k=1}^{\infty} \frac{1}{2^k} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + ...$$

Key points :

A finite series is given by all the terms of a finite sequence, added together.

A infinite series is given by all the terms of an infinite sequence, added together.

Definition

An expression of the form

$$a_1 + a_2 + a_3 + ... + a_n + ...$$

where each a_n is a real number, is called an infinite series of real numbers and is denoted by $\sum_{n=1}^{\infty} a_n$ or $\sum a_n$. a_n is called the nth term of the series $\Sigma\, a_n$.

Example.

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots$$

Example. $\sum_{n=0}^{\infty} \frac{1}{2^n} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$ is an infinite series. The sequence of partial sums looks like :

$$S_0 = 1,\ S_1 = \frac{3}{2},\ S_2 = \frac{7}{4},\ S_3 = \frac{15}{8}, \ldots$$

We saw above that this sequence converges to 2, so

$$\sum_{n=0}^{\infty} \frac{1}{2^n} = \lim S_n = 2.$$

Example. The number n!, read n factorial, is defined as the product of the first n positive integers;

$n! = 1 \cdot 2 \cdots n$

$\langle n! \rangle$ is an important sequence. Its first few terms are

1, 2, 6, 24, 120, 720, ...

By convention, 0! is defined by 0! = 1.

- $1+1+1+1+1+1+\ldots = \sum_{n=1}^{\infty} 1$
- $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots = \sum_{n=1}^{\infty}\frac{1}{n}$
- $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots = \sum_{n=1}^{\infty}\frac{1}{2^n}$
- $1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\ldots = \sum_{n=1}^{\infty}\frac{1}{n^2}$
- $1-1+1-1+1-1+\ldots = \sum_{n=1}^{\infty}(-1)^n$
- $\mathrm{Sin}(x) = \frac{x^1}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \ldots$
- $\mathrm{Cos}(x) = \frac{x^0}{0!} - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \ldots$
- $e^x = \frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$

Sequence of Partial Sums of Series

Recall : A sequence $\{a_n\}$ is a function which assigns a real number a_n to each natural number n : 1, 2, 3, 4, 5, ..., i.e., a sequence is an ordered list of real numbers : $a_1, a_2, a_3, a_4, a_5, \ldots,$

Example : $\left\{\frac{2^{n-1}}{n+7}\right\}$ generates the sequence $\frac{1}{8}, \frac{2}{9}, \frac{4}{10}, \frac{8}{11}, \frac{16}{12}, \ldots$.

Definition : An infinite series $\sum_{n=1}^{\infty} a_n$ is the sum of the numbers in the sequence $\{a_n\}$, i.e., $\sum_{n=1}^{\infty} a_n$ $= a_1 + a_2 + a_3 + a_4 + a_5 + \dots$.

Example : $\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{1}{1}+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+\dots$.

We need a more precise definition of an infinite series $\sum_{n=1}^{\infty} a_n$. Begin by constructing a new sequence of partial sums by letting (This step by step by step process will be called the Sequence of Partial Sums Test for the infinite series $\sum_{n=1}^{\infty} a_n$.)

$s_1 = a_1,$
$s_2 = a_1 + a_2,$
$s_3 = a_1 + a_2 + a_3,$
$s_4 = a_1 + a_2 + a_3 + a_4,$
$s_n = a_1 + a_2 + a_3 + a_4 + \dots + a_n.$

We can now say that the value of the infinite series is precisely the value of the limit of its sequence of partial sums, i.e.,

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + a_4 + \dots$$

$$= \lim_{n\to\infty}(a_1 + a_2 + a_3 + a_4 + \dots + a_n)$$

$$= \lim_{n\to\infty} s_n .$$

Examples of Partial Sums

For the sequence 1,1,1,1,..., we have $\sum_{n=1}^{N} 1 = N \to \infty$. Thus , $\sum_{n=1}^{\infty} 1$ is divergent.

For the sequence 1,-1,1,-1,..., we have $S_1 = 1$, $S_2 = 0$, $S_3 = 1$, etc. In general , $S_N = 1$ for N odd and $S_N = 0$ for N even. Thus, $\sum_{n=1}^{\infty}(-1)^n$ is oscillatory.

The Harmonic Series

If one computes the partial sums for $\sum_{n=1}^{\infty}\frac{1}{n}$ one finds $S_1=1$, $S_2 = \frac{3}{2} = 1.5$, $S_3 = \frac{11}{6} \approx 1.87$, $S_{10} \approx 2.93$, $S_{20} \approx 3.40$, $S_{1000} \approx 7.49$, $S_{100,000} \approx 12.09$. In fact, $S_N \to \infty$ So that $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges.

If one computes the partial sums for $\sum_{n=1}^{\infty}\frac{1}{n^2}$, then one obtains $S_1 = 1$, $S_2 = \frac{5}{4} = 1.25$, $S_3 = \frac{49}{36} \approx 1.36$, $S_{10} \approx 1.55$, $S_{100} \approx 1.63$, $S_{1000} \approx 1.64$.

In fact, $\sum_{n=1}^{\infty}\frac{1}{n^2} = \varsigma(2) = \frac{\pi^2}{6} \approx 1.644934068$

Definitions

- A series $\Sigma\, a_n$ is said to be **convergent,** if the sequence $\langle S_n \rangle$ of partial sums of $\Sigma\, a_n$ is convergent. If $\lim\limits_{n \to \infty} S_n = S$, then S is called the sum of the series $\Sigma\, a_n$. We write $S = \sum\limits_{n=1}^{\infty} a_n$.
- The series $\Sigma\, a_n$ is said to be **divergent,** if the sequence $\langle S_n \rangle$ of partial sums of $\Sigma\, a_n$ is divergent,.
- The series $\Sigma\, a_n$ is said to **oscillate,** if the sequence $\langle S_n \rangle$ of partial sums of $\Sigma\, a_n$ oscillates.

Example : (Geometric series) The series $1 + r + r^2 + r^3 + \ldots$ $(r > 0)$ converges if $r < 1$ and diverges if $r > 1$.

Convergence of a Geometric Series

a geometric series with ratio r diverges if $|r| \geq 1$. If $0 < |r| < 1$, then the series converges to the sum

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}, 0 < |r| < 1$$

Proof : It is easy to see that the series oscillates if $r = \pm 1$. then $s_n = a + ar + ar^2 + \ldots + ar^{n-1}$. multiplication by r yields

$rS_n = ar + ar^2 + \ldots + ar^n$.

Subtracting the second equation from the first produces $S_n - rS_n = a - ar^2$. Therefore.

$S_n(1 - r) = a(1 - r^n)$. and the partial sum is

$$S_n = a\frac{(1-r^n)}{1-r}.$$

If $0 < |r| < 1$, it follows that $r^n \to 0$ as $n \to \infty$, and you obtain

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left[\frac{a}{1-r}(1-r^n)\right] = \frac{a}{1-r}\left[\lim_{n\to\infty}(1-r^n)\right] = \frac{a}{1-r}$$

which means that series converges and its sum is $a/(1 - r)$. It is left to you to show that the series diverges if $|r| > 1$.

Example can be given, such as $a_n = n$ is an unbounded sequence with no limit point and $a_n = 1$, if n is even; $a_n = n$, if n is odd is an unbounded sequence with a limit point 1.

Example : The series $\Sigma\, (-1)^{n-1}$ oscillates.

Example : Perform the following index shifts.

(a) Write $\sum\limits_{n=1}^{\infty} ar^{n-1}$ as a series that starts at $n = 0$.

(b) Write $\sum\limits_{n=1}^{\infty} \frac{n^2}{1-3^{n-1}}$ as a series that starts at $n = 3$.

Solution.

(a) In this case we need to decrease the initial value by 1 and so the n in the term must increase by 1 as well.

$$\sum_{n=1}^{\infty} ar^{n-1} = \sum_{n=0}^{\infty} ar^{(n+1)-1} = \sum_{n=0}^{\infty} ar^n$$

(b) For this problem we want to increase the initial value by 2 and so all the n's in the series terms must decrease by 2.

$$\sum_{n=1}^{\infty}\frac{n^2}{1-3^{n+1}}=\sum_{n=3}^{\infty}\frac{(n-2)^2}{1-3^{(n-2)+1}}=\sum_{n=3}^{\infty}\frac{(n-2)^2}{1-3^{n-1}}$$

Note : The nature of a series is not affected by the addition, alteration or omission of a finite number of terms or by multiplication of all the terms by a fixed non-zero number.

(a) $\sum(u_n \pm v_n) = s \pm t$

(b) $\sum ku_n = ks, k \in \mathbf{R}$

Example : Convergent and Divergent Geometric Series

a. The geometric series

$$\sum_{n\to n}^{\infty} \frac{3}{2^n}=\sum_{n\to n}^{\infty} 3\left(\frac{1}{2}\right)^n$$

$$= 3(1) + 3\left(\frac{1}{2}\right)+3\left(\frac{1}{2}\right)^2+\dots.$$

has a ratio of $r = \frac{1}{2}$ with $a = 3$, Because $0 < |r| < 1$, the series converges and its sum is

$$S = \frac{a}{1-r}=\frac{3}{1-(1/2)}=6.$$

b. The geometric series

$$\sum_{n\to 0}^{\infty}\left(\frac{3}{2}\right)^n=1+\frac{3}{2}+\frac{9}{4}+\frac{27}{8}+\dots.$$

has a ratio of $r = \frac{3}{2}$. Because $|r| \geq 1$. the series diverge.

Example : A Geometric Series for a Repeating Decimal

Use a geometric series to write $0.\overline{08}$ as the ratio of two integers.

Solution. For the repeating decimal $0.\overline{08}$, you can write

$$0.080808\ldots = \frac{8}{10^2}+\frac{8}{10^4}+\frac{8}{10^6}+\frac{8}{10^8}+\ldots$$

$$= \sum_{n\to 0}^{\infty}\left(\frac{8}{10^2}\right)\left(\frac{1}{10^2}\right)^n.$$

For this series, you have $a = 8/10^2$ and $r = 1/10^2$. So.

$$0.080808\ldots = \frac{a}{1-r}=\frac{8/10^2}{1-(1/10^2)}=\frac{8}{99}.$$

Theorem : Limit of nth Term of a Convergent Series

If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

Proof : Assume that

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} S_n = L$$

Then, because $S_n = S_{n-1} + a_n$ and

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} S_{n-1} = L$$

it follows that

$$L = \lim_{n\to\infty} S_n = \lim_{n\to\infty} (S_{n-1} + a_n)$$

$$= \lim_{n\to\infty} S_{n-1} + \lim_{n\to\infty} a_n$$

$$= L + \lim_{n\to\infty} a_n$$

which implies that $\{a_n\}$ converges to 0.

Theorem : nth Term Test for Divergence

If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges.

Example : Using the nth Term Test for Divergence

a. For the series $\sum_{n=0}^{\infty} 2^n$. we have

$$\lim_{n\to\infty} 2^n = \infty.$$

So, the limit of the nth term is not 0 and the series diverges.

b. For the series $\sum_{n=1}^{\infty} \frac{n!}{2n!+1}, \lim_{n\to\infty} a_n = \frac{1}{2}.$

$$\lim_{n\to\infty} \frac{n!}{2n!+1} = \frac{1}{2}.$$

So, the limit of the nth term is not 0 and the series diverges.

c. For the series $\sum_{n=1}^{\infty} \frac{1}{n}$, we have

$$\lim_{n\to\infty} \frac{1}{n} = 0.$$

Because the limit of the nth term is 0. The nth-Term Test for Divergence does not apply and you can draw no conclusion about convergence or divergence.

2. TESTS OF CONVERGENCE OF SERIES

Some important comparison series

1. Geometric series

$$\sum u_n = 1 + r + r^2 + r^3 + \ldots + r^{n-1} + r^n + \ldots$$

(i) is convergent, when $|r| < 1$

(ii) is divergent, when $r \geq 1$

(iii) is oscillating finitely, when $r = -1$

(iv) is oscillating infinitely, when $r < -1$

2. Hyperharmonic series (or p-series)

$$\sum \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \ldots + \frac{1}{n^p} + \ldots$$

(i) is convergent when $p > 1$

(ii) is divergent, when $p \leq 1$

3. The Auxiliary Series

The series $1 + \frac{1}{2(\log 2)^p} + \frac{1}{3(\log 3)^p} + \ldots + \frac{1}{n(\log n)^p} + \ldots$ is convergent if $p > 1$ and divergent if $p \leq 1$.

2.1 Comparison Test

Test I. First Comparison Test

Let $\Sigma\, u_n$ and $\Sigma\, v_n$ be two positive term series such that

$$u_n \leq k\, v_n \quad \forall \quad n \geq m. \qquad \ldots(1)$$

(k being a fixed positive number and m a fixed positive integers)

Then (i) $\Sigma\, v_n$ converges $\Rightarrow$ $\Sigma\, u_n$ converges.

(ii) $\Sigma\, u_n$ diverges $\Rightarrow$ $\Sigma\, u_n$ diverges.

Proof : Let $\langle S_n \rangle$ and $\langle T_n \rangle$ be the sequence of partial sums of the series $\Sigma\, u_n$ and $\Sigma\, v_n$, respectively. For $n \geq m$, we have

$$S_n - S_m = (u_1 + u_2 + \ldots + u_m + u_{m+1} + u_{m+2} + \ldots + u_n) - (u_1 + u_2 + \ldots + u_m)$$

or $S_n - S_m = u_{m+1} + u_{m+2} + \ldots + u_n$. $\qquad \ldots(2)$

Similarly, $T_n - T_m = v_{m+1} + v_{m+2} + \ldots + v_n$. $\qquad \ldots(3)$

From (1) and (2), we obtain

$$S_n - S_m \leq k\,(v_{m+1} + v_{m+2} + \ldots v_n)$$

or $S_n - S_m \leq k\,(T_n - T_m)$, using (3)

or $S_n \leq k\, T_n + a$, $\qquad \ldots(4)$

where $a = S_m - k\, T_m$ is a fixed number.

(i) Suppose $\Sigma\, v_n$ converges.

The sequence $\langle T_n \rangle$ of partial sums of $\Sigma\, v_n$ is bounded above i.e., there exists a positive real number t such that

$$T_n \leq t \quad \forall \quad n. \qquad \ldots(5)$$

From (4) and (5),

$$S_n \leq kt + a \quad \forall \quad n.$$

Thus the sequence $\langle S_n \rangle$ of partial sums of $\Sigma\, u_n$ is bounded above and so $\Sigma\, u_n$ is convergent.

(ii) Suppose $\Sigma\, u_n$ diverges.

Then $\lim_{n\to\infty} S_n = +\infty$. ...(6)

From (4) and (6), we obtain

$$T_n \geq \frac{1}{k}(S_n - a) \Rightarrow \lim_{n\to\infty} T_n = +\infty \quad (\because k > 0).$$

It follows that the sequence $\langle T_n \rangle$ of partial sums of the series $\Sigma\, u_n$ diverges and so $\Sigma\, v_n$ diverges.

Example : Test for convergence the series :

(i) $\sum_{n=2}^{\infty} \frac{1}{\log n}$, (ii) $\sum_{n=2}^{\infty} \frac{1}{n^2 \log n}$

Solution : (i) We know $\log n < n$ for all $n \geq 2$.

$$\therefore \quad \frac{1}{\log n} > \frac{1}{n} \quad \forall \quad n \geq 2 \quad \text{or} \quad \frac{1}{n} < \frac{1}{\log n} \quad \forall \quad n \geq 2.$$

Since $\sum \frac{1}{n}$ diverges, so by First Comparison Test, $\sum_{n=2}^{\infty} \frac{1}{\log n}$ diverges.

(ii) We know $\frac{1}{n^2 \log n} < \frac{1}{n^2} \quad \forall\, n \geq 3$.

Since $\sum \frac{1}{n}$ converges, so by First Comparison Test, $\sum_{n=2}^{\infty} \frac{1}{n^2 \log n}$ converges

Example : Test for the convergence of the series :

$$1 + \frac{1}{2!} + \frac{1}{3!} + \ldots + \frac{1}{n!} + \ldots .$$

Solution : We that $n! \geq 2^{n-1} \quad \forall \quad n \geq 2$.

$$\therefore \quad \frac{1}{n!} \leq \frac{1}{2^{n-1}} \quad \forall\, n \geq 2.$$

Now $\sum_{n=1}^{\infty} \frac{1}{2^{n-1}} = 1 + \frac{1}{2} + \frac{1}{2^2} + \ldots$ being a geometric series with common ratio $\frac{1}{2} < 1$ is convergent. Hence, by First Comparison Test, $\sum \frac{1}{n!}$ is convergent.

Test II. Second Comparison Test

If $\Sigma\, u_n$ and $\Sigma\, v_n$ are two positive term series such that

$$\frac{u_n}{u_{n+1}} \geq \frac{v_n}{v_{n+1}} \quad \forall \quad n \geq m, \qquad \text{...(1)}$$

Then (i) $\Sigma\, v_n$ converges $\Rightarrow$ $\Sigma\, u_n$ converges,

(ii) $\Sigma\, u_n$ diverges $\Rightarrow$ $\Sigma\, v_n$ diverges,

Proof : For $n \geq m$, we have

$$\frac{u_m}{u_n} = \frac{u_m}{u_{m+1}} \cdot \frac{u_{m+1}}{u_{m+2}} \cdot \frac{u_{m+2}}{u_{m+3}} \cdots\cdots \frac{u_{n-1}}{u_n}$$

$$\geq \frac{v_m}{v_{m-1}} \cdot \frac{v_{m+1}}{v_{m+2}} \cdot \frac{v_{m+2}}{v_{m+3}} \cdots\cdots \frac{v_{n-1}}{v_n}, \text{ by (1)}$$

$$= \frac{v_m}{v_n}.$$

Thus $\frac{u_m}{u_n} \geq \frac{v_m}{v_n}$ or $\frac{u_n}{u_m} \leq \frac{v_n}{v_m}$ $\forall$ $n \geq m$

$\therefore$ $u_n \leq k\, v_n$ $\forall$ $n \geq m$, ...(2)

where $k = u_m/v_m$ is a fixed positive number.

(i) $\Sigma\, v_n$ converges $\Rightarrow$ $\Sigma\, u_n$ converges,

(ii) $\Sigma\, u_n$ diverges $\Rightarrow$ $\Sigma\, v_n$ diverges.

Test III. Limit Form Test

Let $\Sigma\, u_n$ and $\Sigma\, v_n$ be two positive term series such that

$$\lim_{n\to\infty} \frac{u_n}{v_n} = l, \ (l \text{ is finite and non-zero}).$$

Then $\Sigma\, u_n$ and $\Sigma\, v_n$ converge or diverge together.

Proof : Since $\frac{u_n}{v_n} > 0$ for all n, so $\lim_{n\to\infty} \frac{u_n}{v_n} \geq 0$.

Thus $l > 0$, as $l \neq 0$

Let $\varepsilon > 0$ be some number such that $l - \varepsilon > 0$.

Since $\lim_{n\to\infty} \frac{u_n}{v_n} = l$, so there exists a positive integer m such that

$$\left|\frac{u_n}{v_n} - l\right| < \varepsilon \quad \forall \quad n \geq m$$

or $\quad l - \varepsilon < \frac{u_n}{v_n} < l + \varepsilon\ \forall\ n \geq m$, where $v_n > 0\ \forall$ n.

$\therefore$ $(l - \varepsilon)\, v_n < u_n < (l + \varepsilon)\, v_n$ $\forall$ $n \geq m$. ...(1)

From (1), we consider

$u_n < (l + \varepsilon)\, v_n$ $\forall$ $n \geq m$. ...(2)

(Here $l + \varepsilon$ is a fixed positive number)

Applying First Comparison Test in (2), we obtain

$$\left.\begin{aligned} &\Sigma\, v_n \text{ converges} \Rightarrow \Sigma\, u_n \text{ converges,} \\ &\Sigma\, u_n \text{ diverges} \Rightarrow \Sigma\, v_n \text{ diverges.} \end{aligned}\right\} \quad ...(A)$$

From (1), we consider

$(l - \varepsilon)\, v_n < u_n$ $\forall$ $n \geq m$

or $\quad v_n < ku_n$ $\forall$ $n \geq m$, ...(3)

where $k = \frac{1}{l - \varepsilon} > 0$ is a fixed number.

Applying First Comparison Test in (3), we obtain

$$\left.\begin{aligned} &\Sigma\, u_n \text{ converges} \Rightarrow \Sigma\, v_n \text{ converges,} \\ &\Sigma\, v_n \text{ diverges} \Rightarrow \Sigma\, u_n \text{ diverges.} \end{aligned}\right\} \quad ...(B)$$

From (A) and (B), it follows that

Σu_n converges $\Leftrightarrow$ Σv_n converges,

Σv_n diverges $\Leftrightarrow$ Σv_n diverges.

Hence the two series Σu_n and Σv_n converge or diverge together.

Remark

- In order to apply the Limit form test to a given series $\sum u_n$, we have to select a series $\sum v_n$ (which is usually a p-series) in which the n^{th} term of v_n behaves as u_n, for large values of n.

Example : Test each of the following series for convergence :

(i) $\frac{1}{1.2.3}+\frac{3}{2.3.4}+\frac{5}{3.4.5}+...$ (ii) $\frac{1}{\sqrt{1.2}}+\frac{1}{\sqrt{2.3}}+\frac{1}{\sqrt{3.4}}+...$

Solution :

(i) $\frac{1}{1.2.3}+\frac{3}{2.3.4}+\frac{5}{3.4.5}+...$

The n^{th} term of this series of $u_n = \frac{2n-1}{n(n+1)(n+2)}$.

For large values of n, $u_n \sim \frac{2n}{n.n.n} = \frac{2}{n^2}$.

Let $v_n = \frac{1}{n^2}$. Then $\frac{u_n}{v_n} = \frac{(2n-1)n^2}{n(n+1)(n+2)}$

or $\frac{u_n}{v_n} = \left(\frac{n}{n+1}\right)\left(\frac{2n-1}{n+2}\right) = \frac{1}{1+(1/n)}.\frac{2-(1/n)}{1+(2/n)}$.

Now $\lim\limits_{n\to\infty} \frac{u_n}{v_n} = 2 \neq 0$ and finite.

So Σu_n and Σv_n converge or diverge together (Limit From Test). Since $\Sigma v_n = \Sigma\frac{1}{n^2}$ converges, so Σu_n converges.

(ii) $\frac{1}{\sqrt{1.2}}+\frac{1}{\sqrt{2.3}}+\frac{1}{\sqrt{3.4}}+...$

The nth term is $u_n = \frac{1}{\sqrt{n(n+1)}} \sim \frac{1}{\sqrt{n.n}} = \frac{1}{n}$.

Let $v_n = \frac{1}{n}$. Then $\frac{u_n}{v_n} = \sqrt{\frac{n}{n+1}} = \frac{1}{\sqrt{1+(1/n)}}$.

Now $\lim\limits_{n\to\infty} \frac{u_n}{v_n} = 1 \neq 0$ and finite.

So Σu_n and Σv_n converge or diverge together.

Since $\Sigma v_n = \Sigma\frac{1}{n}$ diverges, so Σu_n diverges.

Example : Test for convergence the series

$$\frac{1}{\sqrt{2}+\sqrt{3}}+\frac{1}{\sqrt{3}+\sqrt{4}}+\frac{1}{\sqrt{4}+\sqrt{5}}+\ldots$$

Solution. We have

$$u_n=\frac{1}{\sqrt{n+1}+\sqrt{n+2}}\sim\frac{1}{\sqrt{n}+\sqrt{n}}=\frac{1}{2\sqrt{n}}.$$

Let $v_n=\frac{1}{\sqrt{n}}$. Then $\frac{u_n}{v_n}=\frac{\sqrt{n}}{\sqrt{n+1}+\sqrt{n+2}}$.

Now $\lim\limits_{n\to\infty}\frac{u_n}{v_n}=\lim\limits_{n\to\infty}\frac{1}{\sqrt{1+(1/n)}+\sqrt{1+(2/n)}}=\frac{1}{\sqrt{2}}$.

So $\Sigma\, u_n$ and $\Sigma\, v_n$ converge or diverge together.

Example : Test for converges the series

$$\frac{1}{3.7}+\frac{1}{4.9}+\frac{1}{5.11}+\frac{1}{6.13}+\ldots$$

Solution. We have

$$u_n=\frac{1}{(n+2)(2n+5)}\sim\frac{1}{n.2n}=\frac{1}{2n^2}.$$

Let $v_n=\frac{1}{n^2}$. Then

$$\frac{u_n}{v_n}=\frac{n}{n+2}.\frac{n}{2n+5}=\frac{1}{1+(2/n)}.\frac{1}{2+(5/n)}.$$

$$\lim_{n\to\infty}\frac{u_n}{v_n}=\frac{1}{2}\neq 0 \text{ and finite.}$$

So $\Sigma\, u_n$ and $\Sigma\, v_n$ converge or diverge together.

Since $\Sigma\, v_n=\Sigma\, 1/n^2$ converges, so the given series $\Sigma\, u_n$ also converges.

Example : Test for convergence the series whose n^{th} term is

$$\frac{\sqrt{n+1}-\sqrt{n-1}}{n}.$$

Solution.

$$u_n=\frac{\sqrt{n+1}-\sqrt{n-1}}{n}=\frac{(\sqrt{n+1}-\sqrt{n-1})(\sqrt{n+1}+\sqrt{n-1})}{n(\sqrt{n+1}+\sqrt{n-1})}$$

or $$u_n=\frac{(n+1)-(n-1)}{n(\sqrt{n+1}+\sqrt{n-1})}$$

$$=\frac{2}{n(\sqrt{n+1}+\sqrt{n-1})}\sim\frac{2}{n(\sqrt{n}+\sqrt{n})}=\frac{1}{n\sqrt{n}}.$$

Let $v_n = \frac{1}{n\sqrt{n}} = \frac{1}{n^{3/2}}$, so that $\Sigma\, v_n$ converges.

Now $\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{2\sqrt{n}}{(\sqrt{n+1}+\sqrt{n-1})}$

$$= \lim_{n\to\infty} \frac{2}{\sqrt{1+(1/n)}+\sqrt{1-(1/n)}} = 1 \neq 0.$$

So $\Sigma\, u_n$ and $\Sigma\, v_n$ converge or diverge together.

Since $\Sigma\, v_n$ converges, so the given series $\Sigma\, u_n$ also converges.

Example : Test for the convergence of the series :

$$\sum_{n=1}^{\infty}(\sqrt{n^3+1}-\sqrt{n^3}),$$

Solution.

$$u_n = \sqrt{n^3+1}-\sqrt{n^3} = \frac{(\sqrt{n^3+1}-\sqrt{n^3})(\sqrt{n^3+1}+\sqrt{n^3})}{\sqrt{n^3+1}+\sqrt{n^3}}$$

or $u_n = \frac{(n^3+1)-n^3}{\sqrt{n^3+1}+\sqrt{n^3}} = \frac{1}{\sqrt{n^3+1}+\sqrt{n^3}} \sim \frac{1}{2\sqrt{n^3}}.$

Let $v_n = \frac{1}{\sqrt{n^3}} = \frac{1}{n^{3/2}}$, so that $\Sigma\, v_n$ converges.

Now $\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{\sqrt{n^3}}{\sqrt{n^3+1}+\sqrt{n^3}}$

$$= \lim_{n\to\infty} \frac{1}{\sqrt{1+(1/n)^3}+1} = \frac{1}{2} \neq 0 \text{ and finite.}$$

So $\Sigma\, u_n$ and $\Sigma\, v_n$ converge or diverge together.

Since $\Sigma\, v_n$ converges, so the given series $\Sigma\, u_n$ converges.

Example : Test for the convergence of the series whose n^{th} term is $\{(n^3 + 1)^{1/3} - n\}$.

Solution. Let $u_n = (n^3 + 1)^{1/3} - n = n\left[\left(1+\frac{1}{n^3}\right)^{1/3} - 1\right]$

or $u_n = n\left[\left(1+\frac{1}{3n^3}-\frac{1}{9n^6}+\ldots\right)-1\right]$

or $u_n = \frac{1}{3n^2}-\frac{1}{9n^5}+\ldots$

Let $v_n = \frac{1}{n^2}$, so that $\Sigma\, v_n$ converges.

Now $\lim\limits_{n\to\infty} \frac{u_n}{v_n} = \frac{1}{3} \neq 0$ and finite.

So $\Sigma\, u_n$ and $\Sigma\, v_n$ converge or diverge together.

Since $\Sigma\, v_n$ converges, so $\Sigma\, u_n$ converges.

Example : Test for convergence the series

$$\frac{\sqrt{2}-1}{3^3-1}+\frac{\sqrt{3}-1}{4^3-1}+\frac{\sqrt{4}-1}{5^3-1}+\ldots$$

Solution. The n^{th} term of the given series is

$$u_n = \frac{\sqrt{n+1}-1}{(n+2)^3-1}$$

Let $\quad v_n = \frac{\sqrt{n}}{n^3} = \frac{1}{n^{5/2}}$

$$\lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\left(\sqrt{\frac{n+1}{n}}-\frac{1}{\sqrt{n}}\right)\lim_{n\to\infty}\frac{1}{\left[\left(\frac{n+2}{n}\right)^3-\frac{1}{n^3}\right]}$$

$$= \lim_{n\to\infty}\sqrt{1+\left(\frac{1}{n}\right)}\cdot\lim_{n\to\infty}\frac{1}{\left[1+\left(\frac{2}{n}\right)\right]^3} = 1.$$

So $\Sigma\, u_n$ and $\Sigma\, v_n$ converge or diverge together.

Since $\Sigma\, v_n = \Sigma\frac{1}{n^{5/2}}$ converges, so the given series converges.

Example : Test the convergence of the series $S\frac{1}{n^{1+(1/n)}}$.

Solution. We have $u_n = \frac{1}{n^{1+(1/n)}}$. Let $v_n = \frac{1}{n}$.

Then $\lim\limits_{n\to\infty}\frac{u_n}{v_n} = \lim\limits_{n\to\infty}\frac{1}{n^{1/n}} = 1.$

Thus the two series $\Sigma\, u_n$ and $\Sigma\, v_n$ converge or diverge together. Since $\Sigma\, v_n = \Sigma\, 1/n$ diverges, so the given series $\Sigma\, u_n$ diverges.

Example : Show that the series $\frac{1.2}{3^2.4^2}+\frac{3.4}{5^2.6^2}+\frac{5.6}{7^2.8^2}+\ldots$ converges.

Solution. The n^{th} term of the given series is

$$u_n = \frac{(2n-1)\,.\,2n}{(2n+1)^2(2n+2)^2} \sim \frac{1}{n^2}$$ Let $v_n = \frac{1}{n^2}$. Then

$$\frac{u_n}{v_n} = \frac{2n^3(2n-1)}{(2n+1)^2(2n+2)^2} = \frac{4-(2/n)}{(2+1/n)^2(2+2/n)^2}.$$

$\therefore \quad \lim_{n\to\infty} \frac{u_n}{v_n} = 1$, so that $\Sigma\, u_n$ and $\Sigma\, v_n$ converge or diverge together.

Since $\Sigma\, v_n = \Sigma\, 1/n^2$ converges, so the given series $\Sigma\, u_n$ also converges.

Example : Show that the series $1 + a + b + a^2 + b^2 + a^3 + b^3 + \ldots,\ 0 < a < b < 1$ is convergent.

Solution. Since $0 < a < b < 1,\ a^n < b^n\ \forall\ n \geq 0$.

Since $\Sigma\, b^n$ is a geometric series with common ratio b, $0 < b < 1$; $\Sigma\, b^n$ is convergent. Hence, by First comparison test, the given series $\Sigma\, u_n$ is convergent.

2.2 Ratio Test

Test IV. D'Alembert's Ratio Test

Let $\Sigma\, u_n$ be a positive terms series such that

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = l. \qquad \ldots(1)$$

Then (i) $\Sigma\, u_n$ converges if $l > 1$,

(ii) $\Sigma\, u_n$ diverges if $l < 1$.

Test fails if $l = 1$.

Proof. Case I. Let $l > 1$.

We can choose some $\varepsilon > 0$ such that $l - \varepsilon > 1$ or $\alpha > 1,\ \alpha = l - \varepsilon$.

Using (1), there exists a positive integer m_1 such that

$$\left|\frac{u_n}{u_{n+1}} - l\right| < \varepsilon \quad \forall \quad n \geq m_1$$

$$l - \varepsilon < \frac{u_n}{u_{n-1}} < l + \varepsilon \quad \forall \quad n \geq m_1.$$

Consider

$$\frac{u_n}{u_{n+1}} > (l - \varepsilon) = \alpha \ \forall \ n \geq m_1$$

or $$\frac{u_n}{u_{n+1}} > \frac{\alpha^{n+1}}{\alpha^n} \text{ or } \frac{u_n}{u_{n+1}} > \frac{v_n}{v_{n+1}} \quad \forall \quad n \geq m_1, \qquad \ldots(2)$$

where $\Sigma\, v_n = \Sigma \frac{1}{\alpha^n} = \frac{1}{\alpha} + \frac{1}{\alpha^2} + \ldots$, being a geometric series with common ratio $\frac{1}{\alpha} < 1$, is convergent.

Hence $\Sigma\, u_n$ is convergent (by Second Comparison Test is applied in (2)).

Case II. Let $l < 1$.

We can choose another $\varepsilon > 0$ such that $l + \varepsilon < 1$

$$\beta < 1,\ \beta = l + \varepsilon.$$

Using (1), there exists a positive integer m_2 such that

$$\left|\frac{u_n}{u_{n+1}} < l\right| < \varepsilon \quad \forall \quad n \geq m_2$$

or $$l - \varepsilon < \frac{u_n}{u_{n+1}} < l + \varepsilon \quad \forall \quad n \geq m_2.$$

Consider

$$\frac{u_n}{u_{n+1}} < l + \varepsilon = \beta = \frac{\beta^{n+1}}{\beta^n} \quad \forall\ n \geq m_2$$

or $$\frac{v_n}{v_{n+1}} > \frac{u_n}{u_{n+1}} \ \forall \ n \geq m_2, \qquad ...(3)$$

where $\Sigma\, v_n = \Sigma \frac{1}{\beta^n}$, being a geometric series with common ratio $\frac{1}{\beta} > 1$, is divergent. Hence, by Second Comparison Test as applied in (3), $\Sigma\, u_n$ is divergent.

Case III. We shall give examples of two series : one convergent and the other divergent but both satisfying

$$\lim_{n \to \infty} \frac{u_n}{u_{n+1}} = 1.$$

The series $\Sigma\, u_n = \Sigma \frac{1}{n}$ is divergent, but

$$\lim_{n \to \infty} \frac{u_n}{u_{n+1}} = \lim_{n \to \infty} \frac{n+1}{n} = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right) = 1.$$

The series $\Sigma\, u_n = \Sigma \frac{1}{n^2}$ is convergent, but

$$\lim_{n \to \infty} \frac{u_n}{u_{n+1}} = \lim_{n \to \infty} \frac{(n+1)^2}{n^2} = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^2 = 1.$$

Remark 1 : Another equivalent form of Ratio Test is as follows :

If $\Sigma\, u_n$ is a positive term series such that

$$\lim_{n \to \infty} \frac{u_{n+1}}{u_n} = l. \qquad ...(1)$$

Then (i) $\Sigma\, u_n$ is convergent if $l < 1$.

(ii) $\Sigma\, u_n$ is divergent if $l > 1$.

Proof : We have $\lim_{n \to \infty} \frac{u_n}{u_{n+1}} = m$, where $m = \frac{1}{l}$.

Then $\Sigma\, u_n$ converges if $m > 1 \ \Rightarrow \ 1/l > 1 \ \Rightarrow \ l < 1$,

and $\Sigma\, u_n$ converges if $m < 1 \ \Rightarrow \ 1/l < 1 \ \Rightarrow \ l < 1$.

Remark 2 : If $\Sigma\, u_n$ is a positive term series such that

$$\lim_{n \to \infty} \frac{u_n}{u_{n+1}} = \infty, \text{ then } \Sigma\, u_n \text{ is convergent.}$$

Proof : It is given that the sequence $\left\{\frac{u_n}{u_{n+1}}\right\}$ diverges to ∞.

So there exists a positive integer m such that

$$\frac{u_n}{u_{n+1}} > 2 \qquad \forall\ n \geq m.$$

or $$\frac{u_n}{u_{n+1}} > \frac{2^{n+1}}{2^n} \text{ or } \frac{u_n}{u_{n+1}} > \frac{v_n}{v_{n+1}} \qquad \forall\ n \geq m,$$

where $\Sigma\, v_n = \Sigma \frac{1}{2^n}$, being a geometric series with common ratio $\frac{1}{2} < 1$, is convergent. Hence, by Second Comparison Test, $\Sigma\, u_n$ is convergent.

Remark 3 : If $\Sigma\, u_n$ is a positive term series such that

$$\lim_{n \to \infty} \frac{u_{n+1}}{u_n} = 0, \text{ then } \Sigma\, u_n \text{ is convergent.}$$

Proof : For $\varepsilon = \frac{1}{2}$, there exists a positive integer m such that

$$\left|\frac{u_{n+1}}{u_n} - 0\right| < \frac{1}{2} \quad \forall\ n \geq m \quad \text{or} \quad \frac{u_n}{u_{n-1}} > 2 \quad \forall\ n \geq m.$$

The result now follows by Remark 2.

Example : Test for the convergence of the series :

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n!}}.$$

Solution. Let

$$u_n = \frac{1}{\sqrt{n!}}$$

By applying ratio test –

$$\frac{u_{n+1}}{u_n} = \frac{\sqrt{n!}}{\sqrt{(n+1)!}} = \frac{\sqrt{n!}}{\sqrt{(n+1)n!}} = \frac{1}{\sqrt{n+1}}$$

$$\lim_{n \to \infty} \frac{1}{\sqrt{n+1}} = \frac{1}{\infty} = 0 < 1 \Rightarrow \sum u_n \text{ converges.}$$

$$\Rightarrow \qquad \sum u_n = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n!}} \text{ is convergent.}$$

Example : Test for convergence the series

$$\frac{1}{5} + \frac{2!}{5^2} + \frac{3!}{5^3} + \frac{4!}{5^4} + \ldots$$

Solution. We have $u_n = \frac{n!}{5^n}$ and $u_{n+1} = \frac{(n+1)!}{5^{n+1}}$

$\therefore \quad \lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \frac{5.n!}{(n+1)!} = \lim_{n\to\infty} \frac{5}{n+1} = 0 < 1.$

By Ratio Test, the given series diverges.

Example : Test for convergence the series

(i) $\frac{x}{1.3} + \frac{x^2}{2.4} + \frac{x^3}{3.5} + \frac{x^4}{4.6} + \ldots \qquad (x > 0)$

(ii) $\frac{x}{2\sqrt{3}} + \frac{x^2}{3\sqrt{4}} + \frac{x^3}{4\sqrt{5}} + \ldots \qquad (x > 0)$

Solution. (i) $u_n = \frac{x^n}{n(n+2)}$ and $u_{n+1} = \frac{x^{n+1}}{(n+1)(n+3)}$.

$\therefore \quad \lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{1}{x}\left(\frac{n+1}{n}\right)\left(\frac{n+3}{n+2}\right) = \frac{1}{x}.$

By Ratio Test, Σu_n converges of $1/x > 1$ i.e., $x < 1$ and Σu_n diverges if $1/x < 1$ i.e., $x > 1$. The test fails for $x = 1$.

For $x = 1$, $u_n = \frac{1}{n(n+2)} \sim \frac{1}{n^2}$. Let $v_n = \frac{1}{n^2}$.

$\therefore \quad \lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{n}{n+2} = 1.$

So Σu_n and Σv_n converge or diverge together.

Since $\Sigma v_n = \Sigma \frac{1}{n^2}$ converges, so Σu_n converges (for $x = 1$).

Hence the given series converges for $x \le 1$ and diverges for $x > 1$.

(ii) We have

$$u_n = \frac{x^n}{(n+1)\sqrt{n+2}},\ u_{n+1}\ \frac{x^{n+1}}{(n+2)\sqrt{n+3}}.$$

$\therefore \quad \lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{1}{x}\left(\frac{n+2}{n+1}\right)\sqrt{\frac{n+3}{n+2}} = \frac{1}{x}.$

By Ratio Test, the given series converges if $1/x > 1$ i.e., $x < 1$ and diverges if $1/x < 1$ i.e., $x > 1$. The test fails for $x = 1$.

For $x = 1$, $u_n = -\frac{1}{(n+1)\sqrt{n+2}} \sim \frac{1}{n\sqrt{n}}$.

Let $v_n = \frac{1}{n\sqrt{n}}$, so that $\Sigma v_n = \Sigma \frac{1}{n^{3/2}}$ converges.

Now $\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{n}{n+1}\sqrt{\frac{n}{n+2}}$

$$= \lim_{n \to \infty} \frac{1}{\left(1+\frac{1}{n}\right)\sqrt{1+\frac{2}{n}}} = 1.$$

So $\Sigma\, u_n$ and $\Sigma\, v_n$ converge or diverge together. Since $\Sigma\, v_n$ converges, therefore $\Sigma\, u_n$ also converges (for $x = 1$).

Hence the given series converges for $x \le 1$ and diverges for $x > 1$.

Example : Test for the convergence of the series :

(i) $\dfrac{x}{\sqrt{5}} + \dfrac{x^3}{\sqrt{7}} + \dfrac{x^5}{\sqrt{9}} + \ldots$ $\quad (x > 0)$

(ii) $\dfrac{x^2}{2\sqrt{1}} + \dfrac{x^3}{3\sqrt{2}} + \dfrac{x^4}{4\sqrt{3}} + \ldots$ $\quad (x > 0)$

(iii) $\dfrac{x}{2\sqrt{3}} + \dfrac{x^2}{3\sqrt{2}} + \dfrac{x^3}{4\sqrt{5}} + \ldots$ $\quad (x > 0)$

Solution.

(i) $u_n = \dfrac{x^{2n-1}}{\sqrt{2n+3}},\ u_{n+1} = \dfrac{x^{2n+1}}{\sqrt{2n+5}}.$

$$\therefore \quad \lim_{n \to \infty} \frac{u_n}{u_{n+1}} = \frac{1}{x^2} \lim_{n \to \infty} \sqrt{\frac{2n+5}{2n+3}} = \frac{1}{x^2}.$$

By Ratio Test, $\Sigma\, u_n$ converges if $1/x^2 > 1$ i.e., $x^2 < 1$ i.e., $x < 1$ and diverges if $x^2 > 1$ i.e., $x > 1$. The test fails if $x = 1$.

For $x = 1$, $\quad u_n = \dfrac{1}{\sqrt{2n+3}} \sim \dfrac{1}{\sqrt{2n}}.$

Let $\quad v_n = \dfrac{1}{\sqrt{n}}.$

Then $\displaystyle \lim_{n \to \infty} \frac{u_n}{v_n} = \lim_{n \to \infty} \sqrt{\frac{n}{2n+3}} = \frac{1}{2}.$

Since $\Sigma\, v_n = \Sigma \dfrac{1}{\sqrt{n}}$ diverges, so $\Sigma\, u_n$ diverges (for $x = 1$).

Hence the given series converges if $x < 1$ and diverges if $x \ge 1$.

(ii) We have $u_n = \dfrac{x^{n+1}}{(n+1)\sqrt{n}},\ u_{n+1} = \dfrac{x^{n+2}}{(n+2)\sqrt{n+1}}.$

$$\therefore \quad \lim_{n \to \infty} \frac{u_n}{u_{n+1}} = \frac{1}{x} \lim_{n \to \infty} \left(\frac{n+2}{n+1}\right)\sqrt{\frac{n+1}{n}} = \frac{1}{x}.$$

By Ratio Test, $\Sigma\, u_n$ converges if $1/x > 1$ i.e., $x < 1$ and diverges if $x > 1$. The test fails if $x = 1$.

For x = 1, $u_n = \dfrac{1}{(n+1)\sqrt{n}} \sim \dfrac{1}{n\sqrt{n}}$.

Let $v_n = \dfrac{1}{n\sqrt{n}}$.

$$\therefore \quad \lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{(1+1/n)} = 1.$$

Since $\Sigma\, v_n = \Sigma \dfrac{1}{n\sqrt{n}}$ converges, so $\Sigma\, u_n$ converges (for x = 1).

Hence the given series converges if $x \le 1$ and diverges if $x > 1$.

(iii) We have $u_n = \dfrac{x^n}{(n+1)\sqrt{n+2}}$, $u_{n+1} = \dfrac{x^{n+1}}{(n+2)\sqrt{n+3}}$.

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \frac{1}{x} \lim_{n\to\infty} \left(\frac{n+2}{n+1}\right)\sqrt{\frac{n+3}{n+2}} = \frac{1}{x}.$$

By Ratio Test, $\Sigma\, u_n$ converges if $1/x > 1$ i.e., $x < 1$ and diverges if $x > 1$. The test fails if $x = 1$.

For x = 1, $u_n = \dfrac{1}{(n+1)\sqrt{n+2}} \sim \dfrac{1}{n\sqrt{n}}$.

Let $v_n = \dfrac{1}{n\sqrt{n}}$.

As done in part (ii), $\Sigma\, u_n$ converges if x = 1.

Hence the given series converges if $x \le 1$ and diverges if $x > 1$.

Example : Test for convergence the series :

(i) $x + \dfrac{x^3}{3!} + \dfrac{x^5}{5!} + \ldots$ for all positive values of x.

(ii) $\displaystyle\sum_{n=1}^{\infty} \frac{n^n\, x^n}{n!}$ $(x > 0)$.

Solution.

(i) $u_n = \dfrac{x^{2n-1}}{(2n-1)!}$, $u_{n+1} = \dfrac{x^{2n+1}}{(2n+1)!}$.

Now $$\frac{u_n}{u_{n+1}} = \frac{1}{x^2}\frac{(2n+1)!}{(2n-1)!}$$

$$= \frac{1}{x^2}\frac{(2n+1)\,.\,2n\,.(2n-1)!}{(2n-1)!}.$$

$$\therefore \quad \lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{2n(2n+1)}{x^2} = \infty.$$

Hence, by Ratio Test, $\Sigma\, u_n$ converges.

(ii) We have $u_n = \dfrac{n^n x^n}{n!}$ and $u_{n+1} = \dfrac{(n+1)^{n+1}x^{n+1}}{(n+1)!}$

$$\therefore \quad \frac{u_n}{u_{n+1}} = \left(\frac{n+1}{n}\right).\frac{1}{x} = \left(1+\frac{1}{n}\right)\frac{1}{x} \text{ and } \lim_{n\to\infty}\frac{u_n}{u_{n+1}} = \frac{1}{x}.$$

By Ratio test, Σu_n converges if $1/x >$ i.e., $x < 1$ and diverges if $1/x < 1$ i.e., $x > 1$. For $x = 1$, we have

$$u_n = \frac{n^n}{n!} \text{ and } u_{n+1} = \frac{(n+1)^{n-1}}{(n+1)!}$$

$$\therefore \quad \frac{u_n}{u_{n-1}} = \left(\frac{n}{n+1}\right)^n = \frac{1}{(1+1/n)^n} \Rightarrow \lim_{n\to\infty}\frac{u_n}{u_{n+1}} = \frac{1}{e} < 1.$$

So the given series diverges for $x = 1$. Hence the given series converges if $x < 1$ and diverges if $x \geq 1$.

Example : Test for convergence the series :

$$1+\frac{x}{2}+\frac{x^2}{5}+...+\frac{x^n}{n^2+1}+...$$

for all positive values of x.

Solution. Ignoring the first term of the series, we have

$$u_n = \frac{x^n}{n^2+1}, u_{n+1} = \frac{x^{n+1}}{(n+1)^2+1}.$$

$$\therefore \quad \lim_{n\to\infty}\frac{u_n}{u_{n+1}} = \lim_{n\to\infty}\frac{1}{x}.\frac{(n+1)^2+1}{n^2+1} = \frac{1}{x}.$$

By Ratio Test, Σu_n converges if $1/x > 1$ i.e., $x < 1$ and Σu_n diverges if $x > 1$. Ratio Test fails if $x = 1$.

For $x = 1$, $u_n = \dfrac{1}{n^2+1}$.

Clearly, $\dfrac{1}{n^2+1} < \dfrac{1}{n^2} \quad \forall \; n \in \mathbf{N}$.

Since $\Sigma\dfrac{1}{n^2}$ converges, so by First Comparison Test, $\Sigma\dfrac{1}{n^2+1}$ converges.

Hence Σu_n converges for $x \leq 1$ and diverges for $x > 1$.

Example : Test for convergence the series with n^{th} term :

(i) $\dfrac{\sqrt{n}\,x^n}{\sqrt{n^2+1}}$ $(x > 0)$,

(ii) $\sqrt{\dfrac{n-1}{n^3+1}}\,x^n$ $(x > 0)$.

(ii) We have $u_n = \dfrac{n^n x^n}{n!}$ and $u_{n+1} = \dfrac{(n+1)^{n+1}x^{n+1}}{(n+1)!}$

$$\therefore \quad \frac{u_n}{u_{n+1}} = \left(\frac{n+1}{n}\right).\frac{1}{x} = \left(1+\frac{1}{n}\right)\frac{1}{x} \text{ and } \lim_{n\to\infty}\frac{u_n}{u_{n+1}} = \frac{1}{x}.$$

By Ratio test, $\Sigma\, u_n$ converges if $1/x >$ i.e., $x < 1$ and diverges if $1/x < 1$ i.e., $x > 1$. For $x = 1$, we have

$$u_n = \frac{n^n}{n!} \text{ and } u_{n+1} = \frac{(n+1)^{n+1}}{(n+1)!}$$

$$\therefore \quad \frac{u_n}{u_{n+1}} = \left(\frac{n}{n+1}\right)^n = \frac{1}{(1+1/n)^n} \Rightarrow \lim_{n\to\infty}\frac{u_n}{u_{n+1}} = \frac{1}{e} < 1.$$

So the given series diverges for $x = 1$. Hence the given series converges if $x < 1$ and diverges if $x \geq 1$.

Example : Test for convergence the series :

$$1+\frac{x}{2}+\frac{x^2}{5}+\ldots+\frac{x^n}{n^2+1}+\ldots$$

for all positive values of x.

Solution. Ignoring the first term of the series, we have

$$u_n = \frac{x^n}{n^2+1},\; u_{n+1} = \frac{x^{n+1}}{(n+1)^2+1}.$$

$$\therefore \quad \lim_{n\to\infty}\frac{u_n}{u_{n+1}} = \lim_{n\to\infty}\frac{1}{x}.\frac{(n+1)^2+1}{n^2+1} = \frac{1}{x}.$$

By Ratio Test, $\Sigma\, u_n$ converges if $1/x > 1$ i.e., $x < 1$ and $\Sigma\, u_n$ diverges if $x > 1$. Ratio Test fails if $x = 1$.

For $x = 1$, $u_n = \dfrac{1}{n^2+1}$.

Clearly, $\dfrac{1}{n^2+1} < \dfrac{1}{n^2} \quad \forall \; n \in \mathbf{N}$.

Since $\Sigma\dfrac{1}{n^2}$ converges, so by First Comparison Test, $\Sigma\dfrac{1}{n^2+1}$ converges.

Hence Σu_n converges for $x \leq 1$ and diverges for $x > 1$.

Example : Test for convergence the series with n^{th} term :

(i) $\dfrac{\sqrt{n}\,x^n}{\sqrt{n^2+1}}$ $(x > 0)$,

(ii) $\sqrt{\dfrac{n-1}{n^3+1}}\,x^n$ $(x > 0)$.

Solution.

(i) $$u_n = \frac{\sqrt{n}}{\sqrt{n^2+1}}x^n,\ u_{n+1} = \frac{\sqrt{n+1}}{\sqrt{(n+1)^2+1}}x^{n+1}$$

$$\therefore \quad \frac{u_n}{u_{n+1}} = \frac{1}{x}\sqrt{\frac{n}{n+1}}\sqrt{\frac{(n+1)^2+1}{n^2+1}}$$

$$= \frac{1}{x\sqrt{1+1/n}}\sqrt{\frac{(1+1/n)^2+1/n^2}{1+1/n^2}}.$$

$\therefore \quad \lim_{n\to\infty} \frac{u_n}{n_{n+1}} = \frac{1}{x}$. By Ratio Test, the given series converges if $1/x > 1$ i.e., $x < 1$ and diverges if $x > 1$. The test fails if $x = 1$.

For $\quad x = 1,\ u_n = \frac{\sqrt{n}}{\sqrt{n^2+1}} \sim \frac{\sqrt{n}}{\sqrt{n^2}}.$

Let $\quad v_n = \frac{\sqrt{n}}{\sqrt{n^2}} = \frac{1}{\sqrt{n}}$, so that $\Sigma\, v_n$ diverges.

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \sqrt{\frac{n^2}{n^2+1}} = 1.$$

So $\Sigma\, u_n$ and $\Sigma\, v_n$ converge or diverge together.

Since $\Sigma\, v_n$ diverges, so $\Sigma\, u_n$ diverges (for $x = 1$).

Hence the given series converges if $x < 1$ and diverges if $x \geq 1$.

(ii) $$u_n = \sqrt{\frac{n-1}{n^3+1}}x^n,\ u_{n+1} = \sqrt{\frac{n}{(n+1)^3+1}}\ x^{n+1}.$$

$$\therefore \quad \lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{1}{x}\sqrt{\frac{n-1}{n}}\sqrt{\frac{(n+1)^3+1}{n^3+1}} = \frac{1}{x}.$$

By Ratio Test, the given series converges if $1/x >$ i.e., $x < 1$ and diverges if $1/x < 1$ i.e., $x > 1$. The test fails if $x = 1$.

For $\quad x = 1, \quad u_n = \sqrt{\frac{n-1}{n^3+1}} \sim \sqrt{\frac{n}{n^3}} = \frac{1}{n}.$

Let $\quad v_n = \frac{\sqrt{n}}{\sqrt{n^3}} = \frac{1}{n}$, so that $\Sigma\, v_n$ diverges.

$$\therefore \quad \lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \sqrt{\frac{n-1}{n}}\sqrt{\frac{n^3}{n^3+1}} = 1.$$

So $\Sigma\, u_n$ and $\Sigma\, v_n$ converge or diverge together.

Since $\Sigma\, v_n = \Sigma \frac{1}{n}$ diverges, so $\Sigma\, u_n$ diverges (for x = 1).

Hence the given series converges if x < 1 and diverges if $x \geq 1$.

Example : Test for the convergence of the series

$$1+\frac{2^p}{2!}+\frac{3^p}{3!}+\frac{4^p}{4!}+\ldots \quad (p>0).$$

Solution : We have

$$u_n=\frac{n^p}{n!} \text{ and } u_{n+1}=\frac{(n+1)^p}{(n+1)!}.$$

$$\therefore \qquad \frac{u_n}{u_{n+1}}=\frac{n^p}{(n+1)^p}\cdot\frac{(n+1)!}{n!}=(n+1)\cdot\left(\frac{n}{n+1}\right)^p=\frac{n+1}{(1+1/n)^p}.$$

$\lim\limits_{n\to\infty}\frac{u_n}{u_{n+1}}=\infty$. Hence $\Sigma\, u_n$ is convergent, by Ratio Test.

Example : Test for the convergence of the series :

$$1+\frac{2}{5}x+\frac{6}{9}x^2+\frac{14}{17}x^3+\frac{30}{33}x^4+\ldots$$

Solution : Ignoring the first term, we have

$$u_n=\frac{2^{n+1}-2}{2^{n+1}-2}\cdot\frac{2^{n-2}+1}{2^{n+1}+1}=\frac{1}{x}\cdot\frac{2^n-1}{2^{n+1}-1}\cdot\frac{2^{n-2}+1}{2^{n+1}+1}$$

$$\therefore \qquad \frac{u_n}{u_{n+1}}=\frac{1}{x}\cdot\frac{2^{n+1}-2}{2^{n+2}-2}\cdot\frac{2^{n+2}+1}{2^{n+2}+1}=\frac{1}{x}\cdot\frac{2^n-1}{2^{n+1}-1}\cdot\frac{2^{n+2}+1}{2^{n+1}+1}$$

$$=\frac{1}{x}\cdot\frac{1-(1/2)^n}{2-(1/2)^n}\cdot\frac{2+(1/2)^{n-1}}{1+(1/2)^{n-1}}.$$

$$\lim_{n\to\infty}\frac{u_n}{u_{n+1}}=\frac{1}{x}\left(\frac{1-0}{2-0}\right)\left(\frac{2+0}{1+0}\right)=\frac{1}{x}.$$

By Ratio Test, $\Sigma\, u_n$ converges if 1/x > 1 i.e., x < 1 and diverges if x > 1. Ratio Test fails if x = 1.

For $\quad$ x = 1, $\qquad \lim\limits_{n\to\infty} u_n=\lim\limits_{n\to\infty}\frac{2^{n-1}-2}{2^{n+1}+1}=\lim\limits_{n\to\infty}\frac{1-(1/2)^n}{1+(1/2)^{n+1}}$

i.e. $\quad \lim\limits_{n\to\infty} u_n = 1 \neq 0$. So $\Sigma\, u_n$ diverges for x = 1.

Hence $\Sigma\, u_n$ converges if x < 1 and diverges if $x \geq 1$.

2.3 Raabe's Test

Let $\sum u_n$ be a positive term series such that

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right)=l.$$

Then (i) $\sum u_n$ converges if $l > 1$, (ii) $\sum u_n$ diverges if $l < 1$.

The test fails if $l = 1$.

Note : Raabe's test is applied when D'Alembert's ratio test fails

Proof. Case I. Let $l > 1$.

We can choose some $\varepsilon > 0$ such that $l - \varepsilon > 1$ or $\alpha > 1$, $\alpha = l - \varepsilon$.

From (1), there exists a positive integer m such that

$$\left|n\left(\frac{u_n}{u_{n+1}}-1\right)-l\right| < \varepsilon \ \forall \ n \geq m$$

or $$l - \varepsilon < n\left(\frac{u_n}{u_{n+1}}-1\right) < l + \varepsilon = \alpha \ \forall \ n \geq m.$$

Consider

$$n\left(\frac{u_n}{u_{n+1}}-1\right) > l - \varepsilon = \alpha \quad \forall \quad n \geq m$$

or $$nu_n - nu_{n+1} > \alpha \, u_{n+1} \quad \forall \quad n \geq m$$

or $$nu_n - (n + 1)u_{n+1} > (\alpha - 1)u_{n+1} \quad \forall \quad n \geq m.$$

Putting n = m, m + 1, m + 2, ..., n – 1 and adding, we get

$$mu_m - nu_n > (\alpha - 1)(u_{m+1} + u_{m+2} + ... + u_n)$$

or $$(\alpha - 1)(S_n - S_m) < mu_m - nu_n < mu_m,$$

(where $S_n = u_1 + u_2 + ... + u_m + u_{m+1} + ... + u_n$)

Thus $S_n < S_m + \dfrac{m}{\alpha-1}u_m \quad \forall \quad n.$

where $S_m + \dfrac{m}{\alpha-1}u_m$ is a fixed positive number.

It follows that the sequence $\langle S_n \rangle$ is bounded above and so the given series is convergent.

Case II. Let $l < 1$.

We can choose another $\varepsilon > 0$ such that $l + \varepsilon < 1$.

From (1), there exists a positive integer m_1 such that

$$l - \varepsilon < n\left(\frac{u_n}{u_{m+1}}-1\right) < l + \varepsilon \quad \forall \quad n \geq m_1.$$

Since $l + \varepsilon < 1$, $n\left(\dfrac{u_n}{u_{m+1}}-1\right) < 1 \quad \forall \quad n \geq m_1$

or $$\frac{u_n}{u_{m+1}} < 1+\frac{1}{n} = \frac{n+1}{n} \ \forall \ n \geq m_1$$

or $$\frac{v_n}{v_{m+1}} > \frac{u_n}{u_{n+1}} \quad \forall \quad n \geq m_1, \text{ where } v_n = \frac{1}{n}.$$

Since $\Sigma v_n = \Sigma\dfrac{1}{n}$ diverges, so by Second Comparison Test, Σu_n diverges.

Case III. Consider the two series $\sum_{n=1}^{\infty}\frac{1}{n}$ and $\sum_{n=2}^{\infty}\frac{1}{n(\log n)^2}$, where the first series is divergent and the second series is convergent but for both the series

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right)=1.$$

Remark : Raabe's Test is applied when D'Alembert's ratio fails.

When Ratio test fails i.e., when $\lim_{n\to\infty}\frac{u_n}{u_{n+1}} = 1$, Raabe's test may be applied.

Example : Test for convergence the series :

(i) $\sum_{n=1}^{\infty}\frac{3.6.9 \ldots 3n}{7.10.13 \ldots (3n+4)}x^n \ (x>0)$,

(ii) $\sum_{n=1}^{\infty}\frac{2.4.6 \ldots (2n+2)}{3.5.7 \ldots (2n+3)}x^{n-1} \ (x>0)$.

Solution :

(i) $$u_n = \frac{3.6 \ldots 3n}{7.10 \ldots (3n+4)}x^n,$$

$$u_{n+1} = \frac{3.6 \ldots 3n(3n+3)}{7.10 \ldots (3n+4)\,(3n+7)}x^{n+1}$$

$$\therefore \quad \lim_{n\to\infty}\frac{u_n}{u_{n+1}} = \frac{1}{x}\lim_{n\to\infty}\frac{3n+7}{3n+3} = \frac{1}{x}.$$

By Ratio Test, $\Sigma\, u_n$ converges if $1/x > 1$ i.e., $x < 1$ and diverges if $x > 1$. The test fails if $x = 1$. We shall now apply Raabe's Test.

For $x = 1$, $\frac{u_n}{u_{n+1}} = \frac{3n+7}{3n+3}$.

$$\therefore \quad n\left(\frac{u_n}{u_{n+1}}-1\right) = n\left(\frac{3n+7}{3n+3}-1\right) = \frac{4n}{3n+3} = \frac{4}{3+(3/n)}$$

$$\Rightarrow \quad \lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right) = \frac{4}{3} > 1.$$

By Raabe's Test, $\Sigma\, u_n$ converges (for $x = 1$).

Hence the given series converges for $x \le 1$ and diverges for $x > 1$.

(ii) $$u_n = \frac{2.4 \ldots (2n+2)}{3.5 \ldots (2n+3)}x^{n-1},$$

$$u_{n+1} = \frac{2.4 \ldots (2n+2)\,(2n+4)}{3.5 \ldots (2n+3)\,(2n+5)}x^n.$$

$$\therefore \quad \lim_{n\to\infty}\frac{u_n}{u_{n-1}} = \frac{1}{x}\lim_{n\to\infty}\frac{2n+5}{2n+4} = \frac{1}{x}.$$

By Ratio Test, Σu_n converges if $1/x > 1$ i.e., $x < 1$ and diverges if $x > 1$. The test fails if $x = 1$.

For $x = 1$, $\quad \dfrac{u_n}{u_{n+1}} = \dfrac{2n+5}{2n+4}$

now,

$$n\left(\frac{u_n}{u_{n+1}} - 1\right) = n\left(\frac{2n+5}{2n+4} - 1\right) = \frac{n}{2n+4} = \frac{1}{2+(4/n)}.$$

$$\therefore \quad \lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = \frac{1}{2} < 1.$$

By Raabe's Test, Σu_n diverges (for $x = 1$)

Hence the given series converges for $x < 1$ and diverges for $x \geq 1$.

Example : Examine the convergence of the following series :

(i) $\displaystyle\sum_{n=1}^{\infty}\frac{1.3.5\ldots(2n-1)}{2.4.6\ldots 2n}\cdot\frac{x^{2n+1}}{2n+1}, (x > 0)$

(ii) $\displaystyle\sum_{n=1}^{\infty}\frac{1.3.5\ldots(4n-5)(4n-3)}{2.4.6\ldots(4n-4)(4n-2)}\cdot\frac{x^{2n}}{4n}, (x > 0)$

Solution : (i) We have

$$u_n = \frac{1.3\ldots(2n-1)}{2.4\ldots 2n}\cdot\frac{x^{2n+1}}{2n+1},$$

$$u_{n+1} = \frac{1.3\ldots(2n-1)(2n+1)}{2.4\ldots 2n\,(2n+2)}\frac{x^{2n+3}}{2n+3}.$$

$$\therefore \quad \frac{u_n}{u_{n+1}} = \frac{1}{x^2}\frac{(2n+2)(2n+3)}{(2n+1)(2n+1)}, \quad \lim_{n\to\infty}\frac{u_n}{u_{n+1}} = \frac{1}{x^2},$$

By Ratio test, Σu_n converges if $1/x^2 > 1$ i.e., $x < 1$ (as $x > 0$) and diverges if $x > 1$. The test fails for $x = 1$.

For $x = 1$, $\quad \dfrac{u_n}{u_{n+1}} = \dfrac{(2n+2)\,(2n+3)}{(2n+1)^2}$. Now

$$n\left(\frac{u_n}{u_{n+1}} - 1\right) = n\left[\frac{(2n+2)(2n+3)}{(2n+1)^1} - 1\right] = \frac{n(6n+2)}{(2n+1)^2}.$$

$$\therefore \quad \lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = \lim_{n\to\infty}\frac{6+2/n}{(2+1/n)^2} = \frac{6}{4} > 1.$$

$$\therefore \quad \lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \frac{1}{x} \lim_{n\to\infty} \frac{2n+5}{2n+4} = \frac{1}{x}.$$

By Ratio Test, $\Sigma\, u_n$ converges if $1/x > 1$ i.e., $x < 1$ and diverges if $x > 1$. The test fails if $x = 1$.

For $\quad x = 1, \qquad \dfrac{u_n}{u_{n+1}} = \dfrac{2n+5}{2n+4}$

now,

$$n\left(\frac{u_n}{u_{n+1}} - 1\right) = n\left(\frac{2n+5}{2n+4} - 1\right) = \frac{n}{2n+4} = \frac{1}{2+(4/n)}.$$

$$\therefore \quad \lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = \frac{1}{2} < 1.$$

By Raabe's Test, $\Sigma\, u_n$ diverges (for $x = 1$)

Hence the given series converges for $x < 1$ and diverges for $x \geq 1$.

Example : Examine the convergence of the following series :

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{1.3.5 \ldots (2n-1)}{2.4.6 \ldots 2n} \cdot \frac{x^{2n+1}}{2n+1}, \ (x > 0)$

(ii) $\displaystyle\sum_{n=1}^{\infty} \frac{1.3.5 \ldots (4n-5)(4n-3)}{2.4.6 \ldots (4n-4)(4n-2)} \cdot \frac{x^{2n}}{4n}, \ (x > 0)$

Solution : (i) We have

$$u_n = \frac{1.3 \ldots (2n-1)}{2.4 \ldots 2n} \cdot \frac{x^{2n+1}}{2n+1},$$

$$u_{n+1} = \frac{1.3 \ldots (2n-1)(2n+1)}{2.4 \ldots 2n\,(2n+2)} \cdot \frac{x^{2n+3}}{2n+3}.$$

$$\therefore \quad \frac{u_n}{u_{n+1}} = \frac{1}{x^2} \frac{(2n+2)(2n+3)}{(2n+1)(2n+1)}, \quad \lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \frac{1}{x^2},$$

By Ratio test, $\Sigma\, u_n$ converges if $1/x^2 > 1$ i.e., $x < 1$ (as $x > 0$) and diverges if $x > 1$. The test fails for $x = 1$.

For $\quad x = 1, \qquad \dfrac{u_n}{u_{n+1}} = \dfrac{(2n+2)\,(2n+3)}{(2n+1)^2}$. Now

$$n\left(\frac{u_n}{u_{n+1}} - 1\right) = n\left[\frac{(2n+2)(2n+3)}{(2n+1)^1} - 1\right] = \frac{n(6n+2)}{(2n+1)^2}.$$

$$\therefore \quad \lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = \lim_{n\to\infty} \frac{6+2/n}{(2+1/n)^2} = \frac{6}{4} > 1.$$

By Raabe's test, Σu_n converges (for $x = 1$).

Hence the given series converges for $x \leq 1$ and diverges for $x > 1$.

(ii) $$u_n = \frac{1.3 \ldots (4n-3)}{2.4 \ldots (4n-2)} \cdot \frac{x^{2n}}{4n},$$

$$u_{n+1} = \frac{1.3 \ldots (4n-3)(4n+1)}{2.4 \ldots (4n-2)(4n+2)} \frac{x^{2n+2}}{(4n+4)}.$$

$$\therefore \quad \frac{u_n}{u_{n+1}} = \frac{1}{x^2} \frac{(4n+2)(4n+4)}{4n(4n+1)}, \ \lim_{n \to \infty} \frac{u_n}{u_{n+1}} = \frac{1}{x^2}.$$

By Ratio test, Σu_n converges if $1/x^2 > 1$ i.e., $x < 1$ (as $x > 0$) and diverges if $x > 1$. The test fails for $x = 1$.

For $\quad x = 1, \quad \frac{u_n}{u_{n+1}} = \frac{(2n+2)(2n+3)}{(2n+1)^2}$. Now

$$n\left(\frac{u_n}{u_{n+1}} - 1\right) = n\left[\frac{(2n+2)(2n+3)}{(2n+1)^2} - 1\right] = \frac{n(6n+2)}{(2n+1)^2}.$$

$$\therefore \quad \lim_{n \to \infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = \lim_{n \to \infty} \frac{6 + 2/n}{(2 + 1/n)^2} = \frac{6}{4} > 1.$$

By Raabe's Test, Σu_n converges (for $x = 1$).

Hence the given series converges for $x \leq 1$ and diverges for $x > 1$.

(ii) $$u_n = \frac{1.3 \ldots (4n-3)}{2.4 \ldots (4n-2)} \cdot \frac{x^{2n}}{4n},$$

$$u_{n+1} = \frac{1.3 \ldots (4n-3)(4n+1)}{2.4 \ldots (4n-2)(4n+2)} \frac{x^{2n+2}}{(4n+4)}.$$

$$\therefore \quad \frac{u_n}{u_{n+1}} = \frac{1}{x^2} \frac{(4n+2)(4n+4)}{4n(4n+1)}, \ \lim_{n \to \infty} \frac{u_n}{u_{n+1}} = \frac{1}{x^2}.$$

By Ratio test, Σu_n converges if $1/x^2 > 1$ i.e., $x < 1$ (as $x > 0$) and diverges if $x = 1$.

For $\quad x = 1, \quad \frac{u_n}{u_{n+1}} = \frac{(4n+2)(4n+4)}{4n(4n+1)}$. Now

$$n\left(\frac{u_n}{u_{n+1}} - 1\right) = n\left[\frac{(4n+2)(4n+4)}{4n(4n+1)} - 1\right] = \frac{20n+8}{4(4n+1)},$$

and $$\lim_{n \to \infty} n\left(\frac{u_n}{u_{n+1}} + -1\right) = \frac{20}{16} > 1.$$

By Raabe's Test, Σu_n converges (for $x = 1$).

Hence the given series converges for $x \leq 1$ and diverges for $x > 1$.

Example : Test for convergence the series :

$$1+\alpha+\frac{\alpha(\alpha+1)}{1.2}+\frac{\alpha(\alpha+1)(\alpha+2)}{1.2.3}+\ldots\ (\alpha>0)$$

Solution : On ignoring the first term, we have

$$u_n=\frac{\alpha(\alpha+1)\ldots(\alpha+n-1)}{1.2.3\ldots n},\ u_{n+1}=\frac{\alpha(\alpha+1)\ldots(\alpha+n-1)(\alpha+n)}{1.2.3\ldots n(n+1)}$$

$$\therefore \qquad \frac{u_n}{u_{n+1}}=\frac{n+1}{\alpha+n}\Rightarrow\frac{1+(1/n)}{1+(\alpha/n)}\Rightarrow\lim_{n\to\infty}\frac{u_n}{u_{n+1}}=1.$$

Thus Ratio test fails. We shall now apply Raabe's test.

$$n\left(\frac{u_n}{u_{n+1}}-1\right)=n\left(\frac{n+1}{\alpha+n}-1\right)=\frac{n(1-\alpha)}{\alpha+n}=(1-\alpha)\,.\,\frac{1}{1+(\alpha/n)}$$

$$\therefore \qquad \lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right)=1-\alpha<1 \qquad (\because\ \alpha>0)$$

Hence, the given series is divergent.

2.4 De Morgan and Bertrand's Test

If Σu_n be a series of positive terms; such that:

$$\lim_{n\to\infty}\left[\log n\left\{n\left(\frac{u_n}{u_{n+1}}-1\right)-1\right\}\right]=l$$

(i) if $l>1$, Σu_n is convergent

(ii) if $l<1$, Σu_n is divergent

Proof : In which follows, we shall compare Σu_n with the auxiliary series

$$\sum_{n=2}^{\infty}v_n=\sum_{n=2}^{\infty}\frac{1}{n(\log n)^p},$$

which is known to be convergent if $p>1$ and divergent if $p\le 1$.

Now, we have

$$\frac{v_n}{v_{n+1}}=\frac{(n+1)\{\log(n+1)\}^p}{n(\log n)^p}=\frac{n+1}{n}\left[\frac{\log(n+1)}{\log n}\right]^p$$

$$=\left(1+\frac{1}{n}\right)\left[\frac{\log\{n(1+1/n)\}}{\log n}\right]^p=\left(1+\frac{1}{n}\right)\left\{\frac{\log n+\log(1+1/n)}{\log n}\right\}^p$$

$$=\left(1+\frac{1}{n}\right)\left\{1+\frac{1}{\log n}\times\log\left(1+\frac{1}{n}\right)\right\}^p$$

$$=\left(1+\frac{1}{n}\right)\left\{1+\frac{1}{\log n}\left(\frac{1}{n}-\frac{1}{2n^2}+\frac{1}{3n^3}-\ldots\right)\right\}^p$$

$$=\left(1+\frac{1}{n}\right)\left\{1+\frac{1}{\log n}\left(\frac{1}{n}-\frac{1}{2n^2}+\frac{1}{3n^3}-...\right)\right\}^p, \text{ by binomial theorem}$$

$$=1+\frac{1}{n}+\frac{p}{\log n}\left(\frac{1}{n}-\frac{1}{2n^2}+\frac{1}{3n^3}-.....\right)+\frac{p}{n\log n}\left(\frac{1}{n}-\frac{1}{2n^2}+\frac{1}{3n^3}-....\right)$$

Thus $$\frac{v_n}{v_{n+1}}=1+\frac{1}{n}+\frac{p}{n\log n}\left(1+\frac{1}{2n}-\frac{1}{6n^2}+...\right)+... \quad ...(1)$$

Case (i). Let $l > 1$. Choose a number p such that $l \geq p > 1$.

By comparison test of the second type the series Σu_n will be convergent if there exists a positive integer m such that [$n \geq m$

$$\frac{u_n}{u_{n+1}} \geq \frac{v_n}{v_{n+1}}$$

i.e., if $$\frac{u_n}{u_{n+1}} \geq 1+\frac{1}{n}+\frac{p}{n\log n}\left(1+\frac{1}{2n}-\frac{1}{6n^2}+...\right)+..., \text{ by (1)}$$

i.e., if $$\frac{u_n}{u_{n+1}}-1 \geq \frac{1}{n}+\frac{p}{n\log n}\left(1+\frac{1}{2n}-\frac{1}{6n^2}+...\right)+...$$

i.e., if $$n\left(\frac{u_n}{u_{n+1}}-1\right) \geq 1+\frac{p}{n\log n}\left(1+\frac{1}{2n}-\frac{1}{6n^2}+...\right)+...$$

i.e., if $$n\left(\frac{u_n}{u_{n+1}}-1\right)-1 \geq \frac{p}{\log n}\left(1+\frac{1}{2n}-\frac{1}{6n^2}+...\right)+...$$

i.e., if $$\left\{n\left(\frac{u_n}{u_{n-1}}-1\right)-1\right\}\log n \geq p + \text{terms containing n or log n in the denominator} \quad ...(2)$$

Taking limits on both sides of (2) as $n \to \infty$ we get

$$\lim_{n\to\infty}\left[\left\{n\left(\frac{u_n}{u_{n+1}}-1\right)-1\right\}\log n\right] > p$$

$\Rightarrow$ $l > p \Rightarrow l > 1$ as $p > 1$.

Thus, Σu_n converges if $l > 1$.

Case (ii). Proceed as in case (i) yourself.

2.5 Logarithmic Ratio Test

If Σu_n be a series of positive terms; such that :

$$\lim_{n\to\infty}\left[n \log \frac{u_n}{u_{n-1}}\right] = l,$$

(i) if $l > 1$, Σu_n is convergent

(ii) if $l < 1$, Σu_n is divergent

(iii) if $l = 1$, Σu_n may converge or diverge. So test fails.

Proof : Let $l > 1$ and let us choose $\varepsilon > 0$ such that $l - \varepsilon > 1$. Let $l - \varepsilon = \lambda$ so that $l > 1$.

Now $\lim\limits_{n \to \infty} \left(n \log \dfrac{u_n}{u_{n+1}} \right) = l$

$\Rightarrow$ there exists a positive integer m such that

$$l - \varepsilon < n \log \frac{u_n}{u_{n+1}} < l + \varepsilon, \quad \forall\ n \geq m$$

$$\Rightarrow \quad n \log \frac{u_n}{u_{n+1}} < \lambda, \ \forall\ n \geq m$$

$\Rightarrow$ $u_n/u_{n+1} > e^{\lambda/n}$, [$n \geq m$ ...(1)

We now that the sequence $<(1 + 1/n)^n>$ converges to e and hence

$e \geq (1 + 1/n)^n$, [$n \in \mathbf{N}$

$\Rightarrow$ $e^{\lambda/n} \geq (1 + 1/n)^{\lambda}$ ...(2)

From (1) and (2), we have

$$\frac{u_n}{u_{n+1}} > \left(1 + \frac{1}{n}\right)^{\lambda} = \frac{(n+1)^{\lambda}}{n^{\lambda}} = \frac{v_n}{v_{n+1}}, \ \forall\ n \geq m \qquad ...(3)$$

where $v_n = 1/n^{\lambda}$.

Since $\lambda > 1$, so $\Sigma\, v_n$ converges. Then, using comparison test of second type it follows that the given series $\Sigma\, u_n$ also converges.

(ii) Prove as in part (i).

Note 1. The above Logarithmic test is alternative to Raabe's test and should be used when D'Alembert's ratio test fails and when either e occurs in u_n/u_{n+1} or n occurs as an exponent in u_n/u_{n+1}.

Note 2. When Raabe's test fails we may use De Morgan's and Bertrand's test given below. Again, when logarithmic test fails we may use second logarithmic ratio test.

Second Logarithmic Ratio Tests

Let $\Sigma\, u_n$ be a series of positive terms such that

$$\lim_{n \to \infty} \left\{ \left(n \log \frac{u_n}{u_{n+1}} - 1 \right) \log n \right\} = l.$$

Then the series is

(i) convergent if $l > 1$

(ii) divergent if $l < 1$

(iii) no other decision is possible if $l = 1$.

Proof : In what follows, we shall compare $\Sigma\, u_n$ with the auxiliary series

$$\sum_{n=2}^{\infty} v_n = \sum_{n=2}^{\infty} \frac{1}{n(\log n)^p},$$

which is known (refer Art.) to be convergent if $p > 1$ and divergent if $p \leq 1$. Now, we have

$$\frac{v_n}{v_{n+1}} = \frac{(n+1)\{\log(n+1)\}^p}{n(\log n)^p} = \frac{n+1}{n}\left[\frac{\log(n+1)}{\log n}\right]^p$$

$$= \left(1+\frac{1}{n}\right)\left[\frac{\log\{n(1+1/n)\}}{\log n}\right]^p = \left(1+\frac{1}{n}\right)\left\{\frac{\log n+\log(1+1/n)}{\log n}\right\}^p$$

$$= \left(1+\frac{1}{n}\right)\left\{1+\frac{1}{\log n}\log\left(1+\frac{1}{n}\right)\right\}^p = \left(1+\frac{1}{n}\right)\left\{1+\frac{1}{\log n}\left(\frac{1}{n}-\frac{1}{2n^2}+\frac{1}{3n^3}-\ldots\right)\right\}^p$$

$$= \left(1+\frac{1}{n}\right)\left\{1+\frac{p}{\log n}\left(\frac{1}{n}-\frac{1}{2n^2}+\frac{1}{3n^3}-\ldots\right)\right\}, \text{ by binomial theorem}$$

Thus $\dfrac{v_n}{v_{n+1}} = 1+\dfrac{1}{n}+\dfrac{p}{n\log n}+\ldots$ ...(1)

Case (i). Let $l > 1$. Choose a number p such that $l \ge p > 1$.

By comparison test of the second type, the series Σu_n will be convergent if there exists a positive integer m such that [$n \ge m$

$$\frac{u_n}{u_{n+1}} \ge \frac{v_n}{v_{n+1}}$$

i.e., if $\dfrac{u_n}{u_{n+1}} \ge 1+\dfrac{1}{n}+\dfrac{p}{n\log n}+\ldots$, using (1)

i.e., if $\log\dfrac{u_n}{u_{n+1}} \ge \log\left\{1+\left(\dfrac{1}{n}+\dfrac{p}{n\log n}+\ldots\right)\right\}$

i.e., if $\log\dfrac{u_n}{u_{n+1}} \ge \left(\dfrac{1}{n}+\dfrac{p}{n\log n}+\ldots\right)-\dfrac{1}{2}\left(\dfrac{1}{n}+\dfrac{p}{n\log n}+\ldots\right)^3+\ldots$

[$\because$ $\log(1+x) = x - x^2/2 + x^3/3 + \ldots$]

i.e., if $n\log\dfrac{u_n}{u_{n+1}} \ge n\left(\dfrac{1}{n}+\dfrac{p}{n\log n}-\dfrac{1}{2n^2}+\ldots\right)$

i.e., if $n\log\dfrac{u_n}{u_{n+1}} - 1 \ge \dfrac{p}{n\log n}-\dfrac{1}{2n}+\ldots$

i.e., if $\left(n\log\dfrac{u_n}{u_{n+1}}-1\right)\log n \ge p-\dfrac{1}{2}\left(\dfrac{\log n}{n}\right)+\ldots$

$\therefore$ $\lim\limits_{n\to\infty}\left\{\left(n\log\dfrac{u_n}{u_{n+1}}-1\right)\log n\right\} \ge p$, as $\lim\limits_{n\to\infty}\dfrac{\log n}{n} = 0$

or $l \ge p$. But $p > 1$, so $l > 1$.

Hence the given series Σu_n converges if $l > 1$.

(ii) Proceed as in case (i) yourself.

Example : Examine the convergence of the following series :

(i) $$\sum_{n=1}^{\infty}\frac{2\cdot4\cdot6\ldots2n}{1\cdot3\cdot5\ldots(2n+1)}$$

(ii) $$\sum_{n=1}^{\infty}\frac{1\cdot3\cdot5\ldots(4n-5)(4n-3)}{2\cdot4\cdot6\ldots(4n+1)(4n-2)}\frac{x^{2n}}{4n},\ x>0$$

(iii) $$1+\frac{x}{1}+\frac{1}{2}\cdot\frac{x^3}{3}+\frac{1\cdot3}{2\cdot4}\frac{x^5}{5}+\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\frac{x^7}{7}+\ldots.$$

(iv) $$x+\frac{1}{2}\cdot\frac{x^2}{4}+\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\cdot\frac{x^4}{8}+\frac{1\cdot3\cdot5\cdot7\cdot9}{2\cdot4\cdot6\cdot8\cdot10}\cdot\frac{x^6}{12}+\ldots$$

Solution :

(i) Here, we have

$$u_n=\frac{2\cdot4\cdot6\ldots2n}{1\cdot3\cdot5\ldots.(2n+1)},\ u_{n+1}=\frac{2\cdot4\cdot6\ldots2n(2n+2)}{1\cdot3\cdot5\ldots(2n+1)(2n+3)}.\ \text{So}\ \frac{u_n}{u_{n+1}}=\frac{2n+3}{2n+2}$$

Here $$\lim_{n\to\infty}\frac{u_n}{u_{n+1}}=\lim_{n\to\infty}\frac{2+(3/n)}{2+(2/n)}=1.$$

Hence ratio test fails and we now apply Raabe's test.

$$\lim_{n\to\infty}n\left(\frac{u_n}{u_{n+1}}-1\right)=\lim_{n\to\infty}n\left(\frac{2n+3}{2n+2}-1\right)=\lim_{n\to\infty}\frac{n}{2n+2}=\frac{1}{2}<1$$

(ii) $$u_n=\frac{1\cdot3\cdot\ldots.(4n-3)}{2\cdot4\cdot\ldots.(4n-2)}\cdot\frac{x^{2n}}{4n},\ u_{n+1}=\frac{1\cdot3\cdot\ldots.(4n-3)(4n+1)}{2\cdot4\cdot\ldots.(4n-2)(4n+2)}\cdot\frac{x^{2n+2}}{4n+4}$$

$$\therefore\quad \frac{u_n}{u_{n+1}}=\frac{(4n+2)(4n+4)}{4n(4n+1)}\cdot\frac{1}{x^2}\qquad\ldots(1)$$

and $$\lim_{n\to\infty}\frac{u_n}{u_{n+1}}=\lim_{n\to\infty}\frac{(4+2/n)(4+4/n)}{4(4+1/n)}\frac{1}{x^2}=\frac{1}{x^2}$$

Hence by ratio test, $\Sigma\, u_n$ converges if $1/x^2>1$, i.e., $x^2<1$, i.e., $x<1$ (as $x>0$) and diverges if $1/x^2<1$, i.e., $x^2>1$, i.e., $x>1$. The test fails if $x=1$. In that case, from (1), we have

$$\frac{u_n}{u_{n+1}}=\frac{(4n+2)(4n+4)}{4n(4n+1)}$$

$$\therefore\quad n\left(\frac{u_n}{u_{n+1}}-1\right)=n\left\{\frac{(4n+2)(4n+4)}{4n(4n+1)}-1\right\}=\frac{20n+8}{4(4n+1)}$$

and hence

$$\lim_{n\to\infty}n\left(\frac{u_n}{u_{n+1}}-1\right)=\lim_{n\to\infty}\frac{20+8/n}{4(4+1/n)}=\frac{20}{16}>1.$$

Hence, by Raabe's test, $\Sigma\, u_n$ is convergent.

Thus the given series converges if $x \leq 1$ and diverges if $x > 1$.

(iii) Omitting the first term of the given series as it will not change the nature of the series, we obtain

$$u_n = \frac{1\cdot3\cdot5....(2n-1)}{2\cdot4\cdot6....(2n)}\cdot\frac{x^{2n+1}}{2n+1} \text{ and so } u_{n+1} = \frac{1\cdot3\cdot5....(2n-1)(2n+1)}{2\cdot4\cdot6....(2n)(2n+2)}\cdot\frac{x^{2n+3}}{2n+3}$$

$$\therefore \quad \frac{u_n}{u_{n+1}} = \frac{(2n+2)(2n+3)}{(2n+1)^2}\frac{1}{x^2} = \frac{4n^2+10n+6}{4n^2+4n+1}\cdot\frac{1}{x^2} \qquad ...(1)$$

Now $\displaystyle \lim_{n\to\infty}\frac{u_n}{u_{n-1}} = \lim_{n\to\infty}\frac{4+(10/n)+(6/n^2)}{4+(4/n)+(1/n^2)}\cdot\frac{1}{x^2} = \frac{1}{x^2}$

$\therefore$ By ratio test $\Sigma\, u_n$ converges if $1/x^2 > 1$, i.e., if $x^2 < 1$ and diverges if $1/x^2 < 1$, i.e., if $x^2 > 1$. When $1/x^2 = 1$, i.e., $x^2 = 1$, the ratio test fails. We shall now apply Raabe's test.

From (1), for $x^2 = 1$, $\displaystyle \frac{u_n}{u_{n+1}} = \frac{4n^2+10n+6}{4n^2+4n+1}$

$$\therefore \quad \lim_{n\to\infty} n\left(\frac{u_n}{u_{n-1}}-1\right) = \lim_{n\to\infty} n\left(\frac{4n^2+10n+6}{4n^2+4n+1}-1\right) = \lim_{n\to\infty}\frac{6n^2+5n}{4n^2+4n+1} = \frac{3}{2} > 1$$

Hence, by Raabe's test, $\Sigma\, u_n$ converges for $x^2 = 1$.

Thus the given series converges if $x^2 \leq 1$ and diverges if $x^2 > 1$.

(iv) Omitting the first term, the n^{th} term of the resulting series is

$$u_n = \frac{1\cdot3\cdot5\cdot7\cdot9\ ...\ (4n-3)}{2\cdot4\cdot6\cdot8\cdot10\ ...\ (4n-2)}\cdot\frac{x^{2n}}{4n}$$

[Here the nth term of the A.P., 1, 5, 9, ... is $1 + (n-1) \times 4$, i.e., $4n - 3$ and the n^{th} term of the A.P., 2, 6, 10, ... is $2 + (n-1) \times 4$, i.e., $4n - 2$]

and so

$$u_{n+1} = \frac{1\cdot3\cdot5...\ (4n-3)(4n-1)(4n+1)}{2\cdot4\cdot6...\ (4n-2)(4n)(4n+2)}\cdot\frac{x^{2n+2}}{4n+4}$$

$$\therefore \quad \frac{u_n}{u_{n+1}} = \frac{4n\,(4n+2)}{(4n-1)(4n+1)}\times\frac{4n+4}{4n}\times\frac{1}{x^2} = \frac{16n^2+24n+8}{16n^2-1}\cdot\frac{1}{x^2} \qquad ...(1)$$

So $\displaystyle \lim_{n\to\infty}\frac{u_n}{u_{n-1}} = \lim_{n\to\infty}\frac{16+(24/n)+(8/n^2)}{16-(1/n^2)}\cdot\frac{1}{x^2} = \frac{1}{x^2}$

$\therefore$ By ratio test $\Sigma\, u_n$ converges if $1/x^2 > 1$, i.e., if $x^2 < 1$ and diverges if $1/x^2 < 1$, i.e., if $x^2 > 1$. When $1/x^2 = 1$, i.e., $x^2 = 1$, the ratio test fails. We shall now apply Raabe's test.

From (1), for $x^2 = 1$, $\displaystyle \frac{u_n}{u_{n-1}} = \frac{16n^2+24n+8}{16n^2-1}$

$$\therefore \quad \lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right) = \lim_{n\to\infty} n\left(\frac{16n^2+24n+8}{16n^2-1}-1\right) = \lim_{n\to\infty} \frac{24n^2+9n}{16n^2-1} = \frac{3}{2} > 1$$

Hence, by Raabe's test, $\Sigma\ u_n$ converges.

Thus the given series converges if $x^2 \le 1$ and diverges if $x^2 > 1$.

Example : Test the convergence of the following series

(i) $1+\frac{x}{1!}+\frac{2^2\ x^2}{2!}+\frac{3^3\ x^3}{3!}+\ ...$, for $x > 0$

(ii) $\frac{(a+x)}{1!}+\frac{(a+2x)^2}{2!}+\frac{(a+3x)^3}{3!}+...$

Solution : (i) Omitting the first term, we have

$$u_n = \frac{n^n x^n}{n!} \text{ and } u_{n+1} = \frac{(n+1)^{n+1} x^{n+1}}{(n+1)!}$$

$$\text{Now} \quad \frac{u_n}{u_{n+1}} = \frac{(n+1)!}{n!} \times \frac{n^n}{(n+1)^{n+1}} \times \frac{1}{x} = \left(\frac{n}{n+1}\right)^n \times \frac{1}{x} \qquad ...(1)$$

$$\therefore \quad \lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{1}{(1+1/n)^n}.\frac{1}{x} = \frac{1}{ex}, \text{ as } \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$$

By ratio test, $\Sigma\ u_n$ converges if $1/ex > 1$, i.e., if $x < 1/e$ and diverges if $1/ex < 1$, i.e., if $x > 1/e$. When $1/ex = 1$, i.e., $x = 1/e$, the test fails. Since u_n/u_{n+1} will involve e for $x = 1/e$, we shall apply logarithmic test.

For $x = \frac{1}{e}$, from (1) $\quad \frac{u_n}{u_{n+1}} = \left(\frac{n}{n+1}\right)^n \times e$

$$\therefore \quad \lim_{n\to\infty} n\log\frac{u_n}{u_{n+1}} = \lim_{n\to\infty} n\log\left\{\left(\frac{n}{n+1}\right)^n e\right\} = \lim_{n\to\infty} n\log\left\{e\times\left(\frac{n+1}{n}\right)^{-n}\right\}$$

$$= \lim_{n\to\infty} n\ \{\log e - n\log(1+1/n)\}$$

$$= \lim_{n\to\infty} n\left\{1-n\left(\frac{1}{n}-\frac{1}{2n^2}+\frac{1}{3n^3}+....\right)\right\} = \frac{1}{2} < 1$$

Hence, by logarithmic test, $\Sigma\ u_n$ diverges.

Thus the given series converges if $x < 1/e$ and diverges if $x \ge 1/e$.

(ii) Here $u_n = \frac{(a+nx)^n}{n!}$ and $u_{n+1} = \frac{\{a+(n+1)x\}^{n+1}}{(n+1)!}$

$$\text{Now,} \quad \frac{u_n}{u_{n+1}} = \frac{(n+1)!}{n!} \times \frac{(a+nx)^n}{\{a+(n+1)x\}^{n+1}} = \frac{(a+nx)^n(n+1)}{\{a+(n+1)x\}^{n+1}}$$

$$= \frac{n^n x^n (a/nx+1)^n n(1+1/n)}{(n+1)^{n+1} x^{n+1}\{a/(n+1)x+1\}^{n+1}}$$

$$= \frac{1}{x} \cdot \frac{x^{n+1}(1 + a/nx)^n (1+1/n)}{n^{n+1}(1+1/n)^{n+1}\{1 + a/(x+1)x\}^{n+1}}$$

$$= \frac{1}{x} \cdot \frac{\left\{1 + \frac{(a/x)}{n}\right\}^n}{\left(1+\frac{1}{n}\right)^n \left\{1 + \frac{(a/x)}{n+1}\right\}^{n+1}} \qquad ...(1)$$

Hence $\lim\limits_{n\to\infty} \frac{u_n}{u_{n+1}} = \frac{1}{x} \cdot \frac{e^{a/x}}{e \times e^{a/x}} = \frac{1}{ex}$, as $\lim\limits_{n\to\infty}\left(1 + \frac{p}{n}\right)^n = e^p$.

By ratio test, Σu_n converges if $1/ex > 1$, i.e. if $x < 1/e$ and diverges if $1/ex < 1$, if $x > 1/e$

When $1/ex = 1$, i.e., $x = 1/e$, the test fails. We now apply logarithmic test.

For $x = \frac{1}{e}$, from (1), $\frac{u_n}{u_{n+1}} = \frac{e(1 + ea/n)^n}{(1+1/n)^n[1 + ae/(n+1)]^{n+1}}$

$$\log \frac{u_n}{u_{n+1}} = \log e + n \log\left(1 + \frac{ea}{n}\right) - n \log\left(1 + \frac{1}{n}\right) - (n+1) \log\left(1 + \frac{ae}{n+1}\right)$$

$$= 1 + n\left(\frac{ea}{n} - \frac{e^2a^2}{n^2} + \frac{e^3a^3}{n^3}.....\right) - n\left(\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3}.....\right)$$

$$(n-1)\left[\frac{ea}{n+1} - \frac{e^2a^2}{2(n+1)^2} + \frac{e^3a^3}{3(n+1)^3}......\right]$$

$$= \frac{1}{n}\left(\frac{1}{2} - \frac{e^2a^2}{2}\right) + \frac{e^2a^2}{2(n+1)} + \frac{1}{n^2}\left(\frac{e^3a^3}{3} - \frac{1}{3}\right) + ...$$

$$\Rightarrow \quad n \log \frac{u_n}{u_{n+1}} = \frac{1}{2} - \frac{e^2a^2}{2} + \frac{n e^2 a^2}{2(n+1)} + \frac{e^3 a^3 - 1}{3n} + ...$$

$$\Rightarrow \quad \lim_{n\to\infty} n\log\frac{u_n}{u_{n+1}} = \frac{1}{2} - \frac{e^2a^2}{2} + \frac{1}{2} < 1, \quad \text{as} \quad \lim_{n\to\infty}\frac{n}{1+n} = \lim_{n\to\infty}\frac{1}{1+(1/n)} = 1$$

Hence, by logarithmic test, Σu_n diverges.

Thus the given series converges if $x < 1/e$ and diverges if $x \geq 1/e$.

Example : Test for converges the following series :

(i) $\frac{1^2}{2^2} + \frac{1^2 \cdot 3^2}{2^3 \cdot 4^2} + \frac{1^2 \cdot 3^2 \cdot 5^2}{2^2 \cdot 4^2 \cdot 6^2} + ...$

(ii) $\frac{a}{b} + \frac{a(a+1)}{b(b+1)} + \frac{a(a+1)(a+2)}{b(b+1)(b+2)} + ...$

Solution : (i) Here, we have

$$u_n = \frac{1^2 \cdot 3^2 \cdot 5^2....(2n-1)^2}{2^3 \cdot 4^2 \cdot 6^2....(2n)^2}, \quad u_{n+1} = \frac{1^2 \cdot 3^2 \cdot 5^2....(2n-1)^2(2n+1)^2}{2^2 \cdot 4^2 \cdot 6^2....(2n)^2(2n+2)^2}$$

$$\therefore \quad \frac{u_n}{u_{n+1}} = \frac{(2n+2)^2}{(2n+1)^2} = \frac{4n^2+8n+8}{4n^2+4n+1} \qquad ...(1)$$

Here $\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{4+(8/n)+(4/n^2)}{4+(4/n)+(1/n^2)} = 1$

Hence the ratio test fails and we now apply Raabe's test.
Using (1),

$$n\left(\frac{u_n}{u_{n+1}}-1\right) = n\left(\frac{4n^2+8n+4}{4n^2+4n+1}-1\right) = \frac{4n^2+3n}{4n^2+4n+1} \quad \text{...(2)}$$

$$\therefore \quad \lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right) = \lim_{n\to\infty} \frac{4+(3/n)}{4+(4/n)+(1/n^2)} = 1$$

Hence the Raabe's test fails and we now apply De-Morgan's and Bertrand's test.
From (2),

$$n\left(\frac{u_n}{u_{n+1}}-1\right)-1 = \frac{4n^2+3n}{4n^2+4n+1}-1 = \frac{-n-1}{4n^2+4n+1}$$

$$\therefore \quad \left\{n\left(\frac{u_n}{u_{n+1}}-1\right)-1\right\}\log n = -\frac{(n+1)\log n}{4n^2+4n+1} = -\frac{1+(1/n)}{4+(4/n)+(1/n^2)}\cdot\frac{\log n}{n}$$

and so

$$\lim_{n\to\infty} \left\{n\left(\frac{u_n}{u_{n+1}}-1\right)-1\right\}\log n = -\frac{1}{4}\times 0 = 0 < 1, \text{ as } \lim_{n\to\infty}\frac{\log n}{n} = 0$$

Hence, by De-Morgan's and Bertrand's test, $\Sigma\, u_n$ diverges.

(ii) Here $u_n = \frac{a(a+1)(a+2)....(a+n-1)}{b(b+1)(b+2)....(b+n-1)}$,

and $u_{n+1} = \frac{a(a+1)(a+2)....(a+n-1)(a+n)}{b(b+1)(b+2)....(b+n-1)(b+n)}$

$$\therefore \quad \frac{u_n}{u_{n+1}} = \frac{b+n}{a+n} \text{ and so } \lim_{n\to\infty}\frac{b+n}{a+n} = \lim_{n\to\infty}\frac{1+b/n}{1+a/n} = 1 \quad \text{...(1)}$$

Hence the ratio test fails and we now apply Raabe's test.
Using (1),

$$n\left(\frac{u_n}{u_{n+1}}-1\right) = n\left(\frac{b+n}{a+n}-1\right) = \frac{n(b-a)}{a+n} \quad \text{...(2)}$$

$$\therefore \quad \lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right) = \lim_{n\to\infty}\frac{b-a}{(a/n)+1} = b-a$$

Hence, by Raabe's test, $\Sigma\, u_n$ converges if $b - a > 1$ and diverges if $b - a < 1$. When $b - a = 1$. Raabe's test fails and we now apply De-Morgan's and Bertrand's test.

For b – a = 1, from (2), $n\left(\frac{u_n}{u_{n+1}}-1\right)=\frac{n}{n+a}$

$$\therefore \quad n\left(\frac{u_n}{u_{n+1}}-1\right)-1=\frac{n}{n+a}-1=-\frac{a}{n+1}$$

$$\therefore \quad \left\{n\left(\frac{u_n}{u_{n+1}}-1\right)-1\right\}\log n=-\frac{a\log a}{n+1}=\frac{-a}{1+(1/n)}\times\frac{\log n}{n}$$

So $\lim_{n\to\infty}\left\{n\left(\frac{u_n}{u_{n+1}}-1\right)-1\right\}\log n = -a \times 0 = 0 < 1$, as $\lim_{n\to\infty}\frac{\log n}{n} = 0$

Hence, by De-Morgan's and Bertrand's test, $\Sigma\, u_n$ diverges. Thus the given series converges if $b - a > 1$ and diverges if $b - a \le 1$.

Example : Test for convergence the following series

(i) $1^p + \left(\frac{1}{2}\right)^p + \left(\frac{1\cdot 3}{2\cdot 4}\right)^p + \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^p + \ldots$

(ii) $x + x^{1+1/2} + x^{1+1/2+1/3} + x^{1+1/2+1/3+1/4} + \ldots$

Solution : (i) Omitting the first term, we have

$$u_n=\left[\frac{1\cdot 3\cdot 5\ \ldots\ (2n-1)}{2\cdot 4\cdot 6\ \ldots\ (2n)}\right]^p,\quad u_{n+1}=\left[\frac{1\cdot 3\cdot 5\ldots\ (2n-1)(2n+1)}{2\cdot 4\cdot 6\ldots\ (2n)(2n+2)}\right]^p$$

$$\therefore \quad \frac{u_n}{u_{n+1}}=\left(\frac{2n+2}{2n+1}\right)^p=\left(\frac{1+1/n}{1+1/2n}\right)^p \qquad \ldots(1)$$

Here $\lim_{n\to\infty}\frac{u_n}{u_n}=1$ and so ratio test fails.

We shall now apply logarithmic test, Using (1), we get

$$\log\frac{u_n}{u_{n+1}}=p\log\left(\frac{1+1/n}{1+1/2n}\right)=p\ \{\log(1+1/n)-\log(1+1/2n)\}$$

$$=p\left\{\left(\frac{1}{n}-\frac{1}{2n^2}+\frac{1}{3n^3}+\ldots\right)-\left(\frac{1}{2n}-\frac{1}{2\cdot 2^2 n^2}+\frac{1}{3\cdot 2^3 n^3}+\ldots\right)\right\}$$

$$= p(1/2n - 3/8n^2 + 7/24n^3 + \ldots)$$

$$\therefore \quad n\log\frac{u_n}{u_{n+1}}=p\left(\frac{1}{2}-\frac{3}{8n}+\frac{7}{24n^2}+\ldots\right) \qquad \ldots(2)$$

So $\lim_{n\to\infty} n\log\frac{u_n}{u_{n+1}}=\frac{p}{2}.$

Hence $\Sigma\, u_n$ converges if $p/2 > 1$, i.e., if $p > 2$ and diverges if $p/2 < 1$, i.e., if $p < 2$ and the test fails if $p = 2$.

For p = 2, from (2).

$$n\log\frac{u_n}{u_{n+1}} = 2\left(\frac{1}{2} - \frac{3}{8n} + \frac{7}{24n^2} + \ldots\right) = 1 - \frac{3}{4n} + \frac{7}{12n^2} + \ldots.$$

$$\therefore \quad n\log\frac{u_n}{u_{n+1}} - 1 = -\frac{3}{4n} + \frac{7}{12n^2} + \ldots. = \left(-\frac{3}{4} + \frac{7}{12n} + \ldots.\right)\times\frac{1}{n}$$

$$\therefore \quad \lim_{n\to\infty}\left(n\log\frac{u_n}{u_{n+1}} - 1\right)\log n = \lim_{n\to\infty}\left(-\frac{3}{4} + \frac{7}{12n} + \ldots.\right).\frac{\log n}{n} = -\frac{3}{4}\times 0 = 0 < 1$$

Hence by second logarithmic ratio test Σu_n diverges for p = 2.

Thus the given series converges if p > 1 and diverges if p ≤ 2.

(ii) Here, $u_n = x^{1 + 1/2 + \ldots + 1/n}$, $u_{n+1} = x^{1 + 1/2 + \ldots + 1/n + 1/(n+1)}$

$$\therefore \quad \frac{u_n}{u_{n+1}} = \frac{1}{x^{1/(n+1)}} \text{ and so } \lim_{n\to\infty}\frac{u_n}{u_{n+1}} = \frac{1}{x^0} = 1 \qquad \ldots(1)$$

Here ratio test fails and we now apply logarithmic test.

$$\text{From (1),} \quad n\log\frac{u_n}{u_{n+1}} = n\log\left(\frac{1}{x}\right)^{1/(n+1)} = \frac{n}{n+1}\log\frac{1}{x} = \frac{1}{1+(1/n)}\log\frac{1}{x} \qquad \ldots(2)$$

$$\therefore \quad \lim_{n\to\infty} n\log\frac{u_n}{u_{n+1}} = \log\frac{1}{x}$$

Hence Σu_n converges if log (1/x) > 1, i.e., if 1/x > e, i.e., if x < 1/e and Σu_n diverges if log (1/x) < 1, i.e., if x > 1/e. When x = 1/e, the test fails and we shall now apply second logarithmic test

$$\text{For } x = \frac{1}{e}, \text{ from (2)} \qquad n\log\frac{u_n}{u_{n+1}} = \frac{1}{1+(1/n)}, \text{ as } \log e = 1$$

$$\lim_{n\to\infty}\left(n\log\frac{u_n}{u_{n+1}} - 1\right)\log n = \lim_{n\to\infty}\left(\frac{n}{n+1} - 1\right)\log n = \lim_{n\to\infty}\frac{(-1)}{n+1}\log n$$

$$= \lim_{n\to\infty}\left\{\frac{-1}{1+(1/n)}.\frac{\log n}{n}\right\} = -1\times 0 = 0 < 1$$

Hence Σu_n is divergent for x = 1/e.

Thus, the given series converges if x < 1/e and diverges if x ≥ 1/e

2.6 Gauss Test

Theorem : Let $\sum_{n=1}^{\infty} a_n$ be a series with positive terms. Assuming that there exists a real number p, a real number r > 1 and a real bounded sequence $\{B_n\}_{n=1}^{\infty}$ such that for all n

$$\frac{a_n}{a_{n+1}} = 1 + \frac{p}{n} + \frac{B_n}{n^r}$$

Then the series $\sum_{n=1}^{\infty} a_n$ converges if and only if p > 1.

Almost immediately we see that this test is somehow an improved version of Raabe's test. The only difference is the number r, which in the former test was explicitly set to 2. So Gauss' test is more general, allowing us to decide convergence of more series. Not surprisingly the proof is very similar.

Proof. If we start with p > 1, then for sufficiently large n we have (for negative terms of $\{B_n\}$)

$$n\left(\frac{a_n}{a_{n+1}}-1\right)=p+\frac{B_n}{n^{r-1}}\geq q,\ q\in(1,p)$$

or (for positive terms of $\{B_n\}$)

$$n\left(\frac{a_n}{a_{n+1}}-1\right)=p+\frac{B_n}{n^{r-1}}\geq p$$

Since in both cases

$$n\left(\frac{a_n}{a_{n+1}}-1\right)\geq q>1$$

by using Raabe's test we can conclude that the series $\sum_{n=1}^{\infty} a_n$ converges.

As for the other part of equivalence, we will assume that the series $\sum_{n=1}^{\infty} a_n$ converges and p = 1. By using Bertrand's test we get

$$\ln n\left(n\left(\frac{a_n}{a_{n+1}}-1\right)-1\right)=\ln n\left(n\left(1+\frac{1}{n}+\frac{B_n}{n^r}-1\right)-1\right)=$$

$$=\frac{B_n \ln n}{n^{r-1}}\approx\frac{\ln n}{n^{r-1}}$$

With L'Hopital's rule we get $\frac{\ln n}{n^{r-1}}$ $\to 0$ (as n tends to infinity) and therefore the series $\sum_{n=1}^{\infty} a_n$ diverges. A contradiction.

If the series $\sum_{n=1}^{\infty} a_n$ converges and p < 1 then (for sufficiently large n)

$$n\left(\frac{a_n}{a_{n+1}}-1\right)=p+\frac{B_n}{n^{r-1}}<1$$

and according to Raabe's test the series $\sum_{n=1}^{\infty} a_n$ diverges.

A contradiction. Thus p > 1.

Theorem : Let $\sum_{n=1}^{\infty} a_n$ be a series with positive terms. Assuming that there exist a real number p, a real number p, a real number r > 1 and a real bounded sequence $\{B_n\}$ such that for all n.

$$\frac{a_n}{a_{n-1}}=1+\frac{p}{n}+\frac{B_n}{n^r} \qquad ...(1)$$

The series $\sum_{n=1}^{\infty} a_n$ diverges if and only if $p \leq 1$.

Proof : Let $\sum_{n=1}^{\infty} a_n$ diverges if and only if $p \leq 1$. Then for sufficiently large n.

$$n\left(\frac{a_n}{a_{n+1}}-1\right)=p+\frac{B_n}{n^{r-1}}>q,\ q>1$$

and by Raabe's test the series $\sum_{n=1}^{\infty} a_n$ converges. A contradiction (hence $p \le 1$).

If $p = 1$ then by using Bertrand's test (theorem)

$$\ln n\left(n\left(\frac{a_n}{a_{n+1}}-1\right)-1\right)=\frac{B_n \ln n}{n^{(r-1)}}$$

With L'Hospital's rule we get $\frac{\ln n}{n^{r-1}} \to 0$ (as n tends to infinity) and therefore the series $\sum_{n=1}^{\infty} a_n$ diverges.

If $p < 1$ then for sufficiently large n

$$n\left(\frac{a_n}{a_{n+1}}-1\right)=p+\frac{B_n}{n^{r-1}}<1$$

We finish the proof with Raabe's test.

An example. We want to determine the character of the series $\sum_{n=1}^{\infty}\left(\frac{(2n-1)!!}{(2n)!!}\right)^a$, $a > 0$.

$$\frac{a_n}{a_{n+1}}=\left(\frac{\frac{(2n-1)!!}{(2n)!!}}{\frac{(2n+1)!!}{(2n+2)!!}}\right)^a=\left(\frac{2n+2}{2n+1}\right)^a$$

$$n\left(\frac{a_n}{a_{n+1}}-1\right)=n\left(\left(\frac{2n+2}{2n+1}\right)^2-1\right)=n\left(\left(\frac{1+\frac{1}{n}}{1+\frac{1}{2n}}\right)^2-1\right)$$

With $(x \to \infty)$

$$\left(1+\frac{1}{x}\right)^p=1+\frac{p}{x}+o\left(\frac{1}{x}\right)$$

we get the final result

$$n\left(\frac{a_n}{a_{n+1}}-1\right)=n\left(\frac{1+\frac{a}{n}+o\left(\frac{1}{n}\right)}{1+\frac{a}{2n}+\left(\frac{1}{n}\right)}-1\right)=\frac{a}{2}+o(1)$$

According to Raabe's test, the series converges when $a > 2$ and diverges when $a < 2$ but we get no information when $a = 2$. We try Gauss' test :

First, we try to find p

$$p=\lim_{n\to\infty} n\left(\left(\frac{\frac{(2n-1)!!}{(2n)!!}}{\frac{(2n+1)!!}{(2n+2)!!}}\right)^2-1\right)=\lim_{n\to\infty} n\left(\left(\frac{2n+2}{2n+1}\right)^2-1\right)=1$$

Second, we find A_n

$$A_n = \frac{a_n}{a_{n+2}} - 1 - \frac{p}{n} = \left(\frac{\frac{(2n-1)!!}{(2n)!!}}{\frac{(2n+1)!!}{(2n+2)!!}}\right)^2 = \left(\frac{2n+2}{2n+1}\right)^2 - 1 - \frac{1}{n}$$

$$A_n = \frac{-n-1}{4n^3 + 4n^2 + n}$$

Third, we try to find $r > 1$ such that $B_n = A_n n^r$ is bounded

$$A_n = \frac{-n-1}{4n^3 + 4n^2 + n} = \frac{1}{n^2}\left(\frac{-n^3 - n^2}{4n^3 + 4n^2 + n}\right)$$

If we set $r = 2$ then

$$B_n = A_n n^2 = \frac{-n^3 - n^2}{4n^3 + 4n^2 + n} < \infty$$

And because $p = 1$ the series $\sum_{n=1}^{\infty}\left(\frac{(2n-1)!!}{(2n)!!}\right)^2$ diverges.

Therefore, according to Gauss and Raabe's test, the series $\sum_{n=1}^{\infty}\left(\frac{(2n-1)!!}{(2n)!!}\right)^a$ converges when $a > 2$ and diverges when $a \le 2$.

2.7 Kummer's test

Here comes probably the most powerful test for convergence, since it applies to all series with positive terms.

Theorem : Let $\sum_{n=1}^{\infty} a_n$ be a series with positive terms. Then the series converges if and only if there exist a positive number A, positive numbers p_n and a number $N \in \mathbb{N}$ such that for all $n > N$

$$p_n \frac{a_n}{a_{n+1}} - p_{n+1} \ge A \qquad ...(1)$$

The series diverges if and only if there exist positive numbers p_n such that $\sum \frac{1}{p_n} = \infty$ and a number $N \in \mathbb{N}$ such that for all $n > N$

$$p_n \frac{a_n}{a_{n+1}} - p_{n+1} \le 0 \qquad ...(2)$$

Proof : First we prove the convergence. For the right-to-left implication, we adjust the equation (1)

$$p_n a_n - p_{n+1} a_{n+1} \ge A a_{n+1}$$

With $q_n = \frac{p_n}{A}$ we can write

$$q_n a_n - q_{n+1} a_{n+1} \ge a_{n+1} \qquad ...(3)$$

Since we know the left side of upper inequality, we can construct a sequence $\{B_n\}_{n=1}^{\infty}$ such that

$$q_n a_n - q_{n+1} a_{n+1} = B_{n+1} a_{n+1}, \quad \forall n \ B_n \ge 1$$

The sequence $\{q_n a_n\}_{n=1}^{\infty}$ is positive and decreasing (follows from (3)).

Therefore it has a limit

$$0 \le \lim_{n\to\infty} a_n q_n < q_1 a_1$$

Thus the series

$$\sum_{n=1}^{\infty} a_n B_n = \sum_{n=1}^{\infty} (q_n a_n - q_{n+1} a_{n+1}) = q_1 a_1 - \lim_{n\to\infty} q_n a_n > 0$$

converges. And because $B_n \ge 1$ for all n

$$a_n \le a_n B_n \Rightarrow \sum_{n=1}^{\infty} a_n \le \sum_{n=1}^{\infty} a_n B_n$$

In words, the series $\sum_{n=1}^{\infty} a_n B_n$ converges and creates an upper bound for the series $\sum_{n=1}^{\infty} a_n$, therefore the series $\sum_{n=1}^{\infty} a_n$ must converge as well. This is nothing more then just a first comparison criteria. We showed that finding numbers p_n is equally hard as finding a convergent series $\sum b_n$ which creates an upper bound for the series $\sum a_n$ in the first comparison test.

For left-to-right implication, let $p_1 a_1$ be a positive number. Now we assume the existence of positive monotonous sequence $\{B_n\}_{n=1}^{\infty}$

$$\lim_{n\to\infty} B_n = \infty \qquad ...(4)$$

$$\sum_{n=1}^{\infty} a_n B_n = p_1 a_1 + a_1 B_1$$

Now we shift the index n

$$\sum_{n=1}^{\infty} a_{n+1} B_{n+1} = p_1 a_1$$

We define the sequence $\{p_n a_n\}_{n=1}^{\infty}$ this way

$$p_{n+1} a_{n+1} = p_n a_n - a_{n+1} B_{n+1}$$

where

$$\lim_{n\to\infty} p_n a_n = p_1 a_1 - \lim_{n\to\infty} \sum_{n=1}^{\infty} a_{k+1} B_{k+1} = 0$$

So, using (4) we have (for sufficiently large n)

$$p_n a_n - p_{n+1} a_{n+1} = a_{n+1} B_{n+1} \ge A a_{n+1} \quad \text{where} \quad A > 0$$

And finally

$$p_n \frac{a_n}{a_{n+1}} - p_{n+1} \ge A$$

hence the number p_n are found.

Remark : In fact, the requirement (4) is not necessary as any positive and monotonous sequence $\{B_n\}$ with $\lim_{n\to\infty} B_n = A \ge 1$ is totally sufficient (where A is an arbitrary constant).

For example, let $\sum a_n$ be a convergent series. If we let $B_n = 1$ for all n and we construct numbers p_n using the terms from $\sum a_n$

$$p_{n+1} = p_n \frac{a_n}{a_{n+1}} - 1$$

Kummer's test will confirm the convergence of series $\sum a_n$ (as would first comparison test). Now the divergence part.

Proof : For the right-to-left implication, we have $\sum \frac{1}{p^n} = \infty$ and from (2) we get

$$\frac{a_n}{a_{n+1}} \le \frac{\frac{1}{p_n}}{\frac{1}{p_{n+1}}}$$

We can conclude the divergence of series $\sum_{n=1}^{\infty} a_n$ by using the second comparison test.

To prove left-to-right implication (when $\sum_{n=1}^{\infty} a_n$ is divergent), we can assume, according to the theorem, the existence of positive and monotonous sequence B_n such that

$$\lim_{n\to\infty} B_n = 0 \qquad \text{...(5)}$$

$$\sum_{n=1}^{\infty} a_n B_n = \infty$$

We have

$$\frac{a_n}{a_{n+1}} \le \frac{a_n B_n}{a_{n+1} B_{n+1}} \quad \& \quad p_n = \frac{1}{a_n B_n} \quad \Rightarrow \quad \frac{a_n}{a_{n+1}} \le \frac{\frac{1}{p_n}}{\frac{1}{p_{n+1}}}$$

we get

$$p_n \frac{a_n}{a_{n+1}} - p_{n+1} \le 0$$

It is not difficult to see, that $\sum \frac{1}{p^n} = \infty$ and $p_n > 0$ for all n, thus we found the numbers we were looking for.

Remark : Again, the requirement (5) is not necessary as any positive and non-increasing $\{B_n\}$ with $\lim_{n\to\infty} B_n = A > 0$ will do the trick (as it is the monotony we are interested in).

As in the previous remark, if $\sum a_n$ is a divergent series and if we let $B_n = 1$ for all n then Kummer's test will confirm the divergence.

Remark : To sum it up, Kummer's test is very powerful because it really works for all the series with positive terms. On the other hand, using this test is equally difficult as using the first and second comparison test. The true strength of this test therefore lies in the numbers p_n. That is, the form of this test is a masterpiece, not its contents.

To demonstrate the power of Kummer's test, we show that Raabe's test and Bertrand's test are in fact its corollaries. As for Raabe's test, if we set $p_n = n$, we get

$$\exists A > 0, \exists N \in \mathbb{N}, \ \forall n > N : \ n\frac{a_n}{a_{n+1}} - (n+1) \ge A$$

$$n\left(\frac{a_n}{a_{n+1}}-1\right) \geq 1+A$$ compare with for convergence and

$$\left(\sum \frac{1}{n}=\infty\right),\ \exists N \in \mathbb{N},\ \forall n > N\ :\ n\frac{a_n}{a_{n+1}}-(n+1) \leq 0$$

$$n\left(\frac{a_n}{a_{n+1}}-1\right) \leq 1$$ compare with

for divergence.

We see, that what we can decide with Raabe's test, we can also decide with Kummer's test (with $p_n = n$) and vice versa, thus they are equivalent.

As for Kummer's version of Bertrand's test, we set $p_n = n \ln n$ and put it into the left side of (1):

$$n \ln n\frac{a_n}{a_{n+1}}-(n+1)\ln(n+1)=$$

$$=n \ln n\frac{a_n}{a_{n+1}}-(n+1)\left(\ln n+\ln\left(1+\frac{1}{n}\right)\right)=$$

$$=n \ln n\frac{a_n}{a_{n+1}}-n\ln n-\ln n-\ln\left(1+\frac{1}{n}\right)^{(n+1)}=$$

$$=\ln n\left(n\left(\frac{a_n}{a_{n+1}}-1\right)-1\right)-1-\epsilon(n)$$

so $$n \ln n\frac{a_n}{a_{n+1}}-(n+1)\ln(n+1)=\ln n\left(n\left(\frac{a_n}{a_{n+1}}-1\right)-1\right)-1=\varepsilon(n) \quad ...(6)$$

Where

$$\varepsilon(n)=\ln\left(1+\frac{1}{n}\right)^{(n+1)}-1=(n+1)\left(\frac{1}{n}-\frac{1}{2n^2}+o\left(\frac{1}{n^2}\right)\right)-1=\frac{1}{2n}+o\left(\frac{1}{n}\right)$$

We go back to (1). With (6), if

$\exists A > 0,\ \exists N \in \mathbb{N},\ \forall n > N$:

$$\ln n\left(n\left(\frac{a_n}{a_{n+1}}-1\right)-1\right) \geq 1+A+\varepsilon(n) \quad ...(7)$$

then the series $\sum_{n=1}^{\infty} a_n$ is convergent.

Compare with Bertrand's test :

If $\exists A > 0,\ \exists N \in \mathbb{N},\ \forall n > N$:

$$\ln n\left(n\left(\frac{a_n}{a_{n+1}}-1\right)-1\right) \geq 1+A \quad ...(8)$$

then the series $\sum_{n=1}^{\infty} a_n$ is convergent.

Since $\varepsilon(n)$ can get arbitrarily small as n tends to infinity, we can hide it inside the positive constant A. Thus (7) and (8) are equivalent.

Divergence is a bit different and we will see that in this case, the tests are not equivalent. That is, Kummer's test is slightly stronger. It is because now we have zero as a sharp border, while the constant A from the previous case was quite flexible.

With (2), (6) and $p_n = n \ln n$: if

$$\left(\sum \frac{1}{n \ln n} = \infty\right), \quad \exists N \in \mathbb{N}, \ \forall n > N :$$

$$\ln n\left(n\left(\frac{a_n}{a_{n+1}} - 1\right) - 1\right) - 1 \leq \varepsilon(n)$$

then the series $\sum_{n=1}^{\infty}$ is divergent. Compare with Bertrand's test (see (5)) : If

$$\exists N \in \infty, \quad \forall n > N \ : \ \ln n\left(n\left(\frac{a_n}{a_{n+1}} - 1\right) - 1\right) - 1 \leq \varepsilon(n)$$

then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

Now let's consider the series $\sum_{n=1}^{\infty} a_n$, where $a_n = \dfrac{1}{n \ln n}$. Using Bertrand's test

$$\ln n\left(n\left(\frac{(n+1)\ln(n+1)}{n \ln n} - 1\right) - 1\right) - 1 \leq 0$$

$$(n + 1) \ln (n + 1) - n \ln n - \ln n - 1 = (n + 1) \ln \left(1 + \frac{1}{n}\right) - 1 = \varepsilon(n) \leq 0 \rightarrow \text{false}$$

Using Kummer's test (with $p_n = n \ln n$)

$$n \ln n \frac{(n+1)\ln(n+1)}{n \ln n} - (n+1)\ln(n+1) \leq 0$$

$$0 \leq 0 \quad \rightarrow \quad \text{true}$$

There is infinite number of series that can be decided only with Kummer's version of Bertrand's test, but if we use limit, the tests are equivalent.

2.8 Root Test

Test IX. Cauchy's nth Root Test

Let Σu_n be a positive terms series such that

$$\lim_{n \to \infty} (u_n)^{1/n} = l. \qquad \text{...(1)}$$

Then **(i)** Σu_n converges if $l < 1$, **(ii)** Σu_n diverges if $l > 1$. Test fails if $l = 1$.

Proof : Case I. Let $l < 1$.

We can choose some $\varepsilon > 0$ such that $l + \varepsilon < 1$ or $\alpha < 1$, $\alpha = l + \varepsilon$.

Using (1), there exists a positive integer m_1 such that

$$|(u_n)^{1/n} - l| < \varepsilon \quad \forall \quad n \geq m_1$$

or $$l - \varepsilon < (u_n)^{1/n} < l + \varepsilon \quad \forall \quad n \geq m_1.$$

Consider $$(u_n)^{1/n} < l + \varepsilon = \alpha \ \forall \quad n \geq m_1$$

$$u_n < \alpha^n \quad \forall \quad n \geq m_1. \qquad \text{...(2)}$$

Since $\Sigma \alpha^n = \alpha + \alpha^2 + \ldots$ being a geometric series with common ratio $\alpha < 1$ is convergent, so by **First Comparison Test** as applied in (2), Σu_n is convergent.

Case II. Let $l > 1$.

We can choose another $\varepsilon > 0$ such that $l - \varepsilon > 1$ or $\beta > 1$, $\beta = l - \varepsilon$.

Using (1), there exists a positive integer m_2 such that

$$|(u_n)^{1/n} - l| < \varepsilon \quad \forall \quad n \geq m_2$$

or $$l - \varepsilon < (u_n)^{1/n} < l + \varepsilon \quad \forall \quad n \geq m_2.$$

Consider $$l - \varepsilon < (u_n)^{1/n} \quad \forall \quad n \geq m_2$$

or $$\beta^n < u_n \quad \forall \quad n \geq m_2. \qquad \text{...(3)}$$

Since $\sum \beta^n$ being a geometric series with common ratio $\beta > 1$ is divergent, so by First Comparison Test as applied in (3), $\sum u_n$ is divergent.

Case III. We shall give examples of two series : one convergent and the other divergent, but both satisfying $\lim\limits_{n \to \infty} (u_n)^{1/n} = 1$.

The series $\sum u_n = \sum \frac{1}{n}$ is divergent, but

$$\lim_{n \to \infty} (u_n)^{1/n} = \lim_{n \to \infty} \left(\frac{1}{n}\right)^{1/n} = 1.$$

The series $\sum u_n = \sum \frac{1}{n^2}$ is convergent, but

$$\lim_{n \to \infty} (u_n)^{1/n} = \lim_{n \to \infty} \left(\frac{1}{n^2}\right)^{1/n} = 1.$$

3. CAUCHY'S CONDENSATION TEST

Statement : For a non-increasing sequence $\{a_n\}$ of non-negative real numbers, the series $\sum a_n$ converges if and only if the series $\sum 2^n a_{2^n}$ converges.

Proof : Observe that since $\{a_n\}$ is decreasing :

$$a_1 + (a_2 + a_3) + (a_4 + a_5 + a_6 + a_7) + \ldots + a_N \le a_1 + 2a_2 + 4a_4 + \ldots + 2^N a_{2^N}$$

$$\Rightarrow \quad \sum a_n \le \sum 2^N a_{2^N}$$

$$\sum_{n=2}^{N} 2a_n = a_2 + (a_2 + a_3) + (a_3 + 2a_4 + a_5) + (a_5 + 2a_6 + 2a_7 + 2a_8 + a_9) + \ldots + 2a_N$$

$$\ge a_2 + 2a_4 + 4a_8 + 8a_{16} + \ldots + 2^n a_{2^n}$$

$$\Rightarrow \quad 2\sum a_n \ge \sum 2^N a_{2^N}$$

Thus proved.

Example : Discuss convergence of $\sum_{n=1}^{\infty} \frac{1}{n(\log n)^p}$, $p > 0$

Solution : $\frac{1}{n(\log n)^p}$ is monotone decreasing as log n is increasing.

If $\quad \sum a_n = \sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}, \sum 2^n a_{2^n} = \sum_{n=2}^{\infty} \frac{1}{(\log 2^n)^p} = \frac{1}{\log 2} \sum_{n=2}^{\infty} \frac{1}{n^p}.$

Thus, the series converges iff $p > 1$.

Statements of two important tests for series with arbitrary terms :

Abel's test : If $\{(x_n)\}$ is a convergent monotone sequence and the series $\sum y_n$ is convergent, then the series $\sum x_n y_n$ is also convergent.

Dirichlet's test : If $\{(x_n)\}$ is a decreasing sequence with $\lim x_n = 0$, and if the partial sums $\{(s_n)\}$ of $\sum y_n$ are bounded, then the series $\sum x_n y_n$ is convergent.

4. ALTERNATING SERIES

A series of the form $u_1 - u_2 + u_3 - u_4 + \ldots$, where $u_n > 0\ \forall\ n \in N$ is called an **alternating series** and is denoted by $\sum_{n=1}^{\infty} (-1)^{n-1} u_n$.

We have $\sum_{n=1}^{\infty} (-1)^{n-1} u_n = u_1 - u_2 + u_3 - u_4 + \ldots + (-1)^{n-1} u_n + \ldots$

Example :

$$1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \ldots\ldots.$$

4.1 Leibnitz Test

If an alternating series $\sum_{n=1}^{\infty} (-1)^{n-1} u_n$ satisfies

(i) Each term is numerically less than the proceeding term i.e.

$u_n \le u_{n-1} \quad \forall\ n,$

(ii) $\lim_{n \to \infty} u_n = 0,$

then the series $\Sigma\ (-1)^{n-1} u_n$ converges.

Proof : In order to prove that the given series converges, we shall show that its sequence $\langle S_n \rangle$ of partial sums converges. For this we shall first prove that the subsequences $\langle S_n \rangle$ and $\langle S_{2n+1} \rangle$ both converge to the same limit.

Now $S_{2n} = u_1 - u_2 + u_3 - u_4 + \ldots + u_{2n-1} - u_{2n}$,

and $S_{2n+2} = u_1 - u_2 + u_3 - u_4 + \ldots + u_{2n-1} - u_{2n} + u_{2n+1} - u_{2n+2}$.

$\therefore \quad S_{2n+2} - S_{2n} + u_{2n+1} - u_{2n+2} \ge 0. \qquad (\because\ u_{n+1} \le u_n \quad \forall\ n)$

Thus $\langle S_{2n} \rangle$ is a monotonically increasing sequence.

Again $S_{2n} = u_1 - u_2 + u_3 - u_4 + u_5 - u_6 + \ldots + u_{2n-1} - u_{2n}$

$= u_1 - [(u_2 - u_3) + (u_4 - u_5) + \ldots + (u_{2n-2} - u_{2n-1}) + u_{2n}]$.

Now each term within the bracket is positive, since

$u_{n+1} \le u_n \quad \forall \quad n \quad$ and $\quad u_{2n} > 0$.

$\therefore \quad S_{2n} < u_1 \quad \forall \quad n \quad$ and $\quad$ so $\langle S_{2n} \rangle$ is bounded above.

Since $\langle S_{2n} \rangle$ is monotonically increasing and bounded above, therefore $\langle S_{2n} \rangle$ is convergent.

Let $\lim_{n \to \infty} S_{2n} = l.$ ...(iii)

We shall now show that $\langle S_{2n+1} \rangle$ converges to l. We have

$S_{2n+1} = u_1 - u_2 + u_3 - u_4 + \ldots + u_{2n-1} - u_{2n} + u_{2n+1}$

or $S_{2n+1} = S_{2n} + u_{2n+1} \Rightarrow \lim_{n \to \infty} S_{2n+1} = \lim_{n \to \infty} (S_{2n} + u_{2n+1})$

or $\lim_{n \to \infty} S_{2n+1} = \lim_{n \to \infty} S_{2n} + \lim_{n \to \infty} u_{2n+1} = \lambda + 0$, by (ii) and (iii)

$\therefore \quad \lim_{n \to \infty} S_{2n+1} = l.$

From (iii), (iv), it follows that for any $\varepsilon > 0$, there exists positive integers m_1 and m_2 such that

$| S_{2n} - l | < \varepsilon, \forall \quad n \ge m_1,$ ...(v)

and $| S_{2n+1} - l | < \varepsilon, \forall \quad n \ge m_2.$ ...(vi)

Let $m = \max(m_1, m_2)$ so that $m \geq m_1$, $m \geq m_2$, ...(vii)

From (v), (vi), (vii) ; $| S_n - l | < \varepsilon \ \forall \ n \geq m$

$\Rightarrow$ $\langle S_n \rangle$ converges to l.

Hence $\sum_{n=1}^{\infty}(-1)^{n-1}u_n$ is convergent.

Remark : The alternating series $\Sigma(-1)^{n-1} u_n$ will not be convergent if either (i) $u_{n+1} \not< u_n$, or $\lim_{n\to\infty} u_n \neq 0$.

For Example

The series $1 - \frac{3}{2} + \frac{4}{3} - \frac{5}{4} + \ldots$ is not convergent,

$\because \lim_{n\to\infty} u_n = \lim_{n\to\infty}\left(\frac{n+1}{n}\right) = \lim_{n\to\infty}\left(1+\frac{1}{n}\right) = 1 \neq 0.$

Example : Let $a_n = 1 - \frac{1}{2} + \frac{1}{3} + \ldots + \frac{(-1)^{n+1}}{n}$

This is an alternating series

So we can check the convergence by applying leibnitz test

(i) $u_n \geq 0$

$u_n \leq u_{n-1}$

as $\frac{1}{2} \leq 1$

$\frac{1}{3} < \frac{1}{2}$

and so on

i.e. Each term of a_n is decreasing

(ii) $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{n} = 0$

So by Leibnitz test the given series converges.

Example : Test for convergence the series

(a) $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$ (b) $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{\sqrt{n}} = 1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \ldots$

Solution :

(a) We have

$\frac{1}{2} < 1, \frac{1}{3} < \frac{1}{2}, \frac{1}{4} < \frac{1}{3}$ and so on. Thus

(i) $u_{n+1} < u_n \ \forall \ n$,

(ii) $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{n} = 0.$

Hence, by Leibnitz's Test, the given series is convergent.

(b) The proof of similar. The given series is convergent.

Example : Using the Alternating Series Test

Determine the convergence or divergence of $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$.

Solution : Note that $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{1}{n} = 0$. So, the first condition of Theorem is satisfied. Also note that the second condition of Theorem satisfied because

$$a_{n+1} = \frac{1}{n+1} \le \frac{1}{n} = a_n$$

for all n. So, applying the Alternating Series Test, you can conclude the series converges.

Example : Using the Alternating Series Test

Determine the convergence or divergence of $\sum_{n=1}^{\infty}\frac{n}{(-2)^{n-1}}$.

Solution : To apply the Alternating Series Test, note that, for $n \ge 1$,

$$\frac{1}{2} \le \frac{n}{n+1}$$

$$\frac{2^{n-1}}{2^n} \le \frac{n}{n+1}$$

$$\{n + 1\}2^{n-1} \le n2^n$$

$$\frac{n+1}{2^n} \le \frac{n}{2^{n-1}}.$$

So, $a_{n+1} = (n + 1)/2^n \le n/2^{n-1} = a_n$ for all n. Furthermore, by L'Hopital's Rule,

$$\lim_{n\to\infty}\frac{x}{2^{x-1}} = \lim_{n\to\infty}\frac{1}{2^{x-1}(\ln 2)} = 0 \Rightarrow \lim_{n\to\infty}\frac{n}{2^{n-1}} = 0.$$

Therefore, by the Alternating Series Test, the series converges.

Example : When the Alternating Series Test Does Not Apply

a. The alternating series

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(n+1)}{n} = \frac{2}{1} - \frac{3}{2} + \frac{4}{3} - \frac{5}{4} + \frac{6}{5} - \dots$$

passes the second condition of the Alternating Series Test because $a_{n+1} \le a_n$ for all n. You cannot apply the Alternating. Series Test, however, because the series does not pass the first condition. In fact, the series diverges.

b. The alternating series

$$\frac{2}{1} - \frac{1}{1} + \frac{2}{2} - \frac{1}{2} + \frac{2}{3} - \frac{1}{3} + \frac{2}{4} - \frac{1}{4} + \dots$$

passes the first condition because a_n approaches 0 as $n \to \infty$. You cannot apply the Alternating Series Test, however, because the series does not pass the second condition. To conclude that the series diverges, you can argue that S_{2N} equals the N^{th} partial sum of the divergent harmonic series. This implies that the sequence of partial sums diverges. So, the series diverges.

Example : The alternating harmonic series

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$$

satisfies

(i) $b_{n+1} < b_n$ because $\dfrac{1}{n+1}<\dfrac{1}{n}$

(ii) $\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty}\dfrac{1}{n}=0$

So the series is convergent by the Alternating Series Test.

5. ABSOLUTE AND CONDITIONAL CONVERGENCE

Sometimes series have both positive and negative terms but they are not perfectly alternating like those in the previous section. For example

$$\sum_{n=1}^{\infty}\frac{\sin n}{n^2}\approx\frac{0.841}{1}+\frac{0.909}{4}+\frac{0.141}{9}-\frac{0.757}{16}-\frac{0.279}{25}+\frac{0.657}{36}+\ldots$$

is not alternating but does have both positive and negative terms.

So how do we deal with such series? The answer is to take the absolute value of the terms. This turns the sequence into a non-negative series and now we can apply many of our previous convergence tests. For example if we take the absolute value of the terms in the series above, we get

$$\sum_{n=1}^{\infty}\left|\frac{\sin n}{n^2}\right|.$$

Since $|\sin n| \le 1$, then

$$0<\left|\frac{\sin n}{n^2}\right|<\frac{1}{n^2}.$$

But $\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges by the p-series test ($p = 2 > 1$), so $\sum_{n=1}^{\infty}\left|\frac{\sin n}{n^2}\right|$ converges by comparison.

But what about the original series $\sum_{n=1}^{\infty}\frac{\sin n}{n^2}$? The next theorem provides the answer: The series does converge.

Theorem: (The Absolute Convergence Test). If $\sum_{n=1}^{\infty}|a_n|$ converges so does $\sum_{n=1}^{\infty}a_n$.

Proof. Given $\sum_{n=1}^{\infty}|a_n|$ converges. Define a new series $\sum_{n=1}^{\infty}b_n$, where

$$b_n=a_n+|a_n|=\begin{cases}a_n+a_n=2a_n, & \text{if } a_n\ge 0\\ a_n-a_n=0, & \text{if } a_n<0\end{cases}\ge 0.$$

So $0 \le b_n = a_n + |a_n| \le |a_n| + |a_n| = 2|a_n|$. But $\sum_{n=1}^{\infty}2|a_n|$ converges, hence by direct comparison $\sum_{n=1}^{\infty}b_n$ converges. Therefore

$$\sum_{n=1}^{\infty}a_n=\sum_{n=1}^{\infty}\left[\left(a_n+|a_n|\right)-|a_n|\right]=\sum_{n=1}^{\infty}b_n-|a_n|=\sum_{n=1}^{\infty}b_n-\sum_{n=1}^{\infty}|a_n|$$

converges since it is the difference of two convergent series.

Important Note : The converse is not true. If $\sum_{n=1}^{\infty}a_n$ converges, $\sum_{n=1}^{\infty}|a_n|$ may or may not converge.

For example, the alternating harmonic series $\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$ converges, but if we take the absolute value of the terms, the harmonic series $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges.

This leads to the following definition.

Absolute Convergence

A series Σu_n is said to be **absolutely convergent** if the series $\Sigma |u_n|$ is convergent.

Example :

The series $\sum u_n = 1 - \frac{1}{2} + \frac{1}{2^2} - \frac{1}{2^3} + \dots$ is absolutely convergent, since $\sum |u_n| = 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots$, being a geometric series with common ratio r = $\frac{1}{2} < 1$, is convergent.

Example : Determine whether $\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$ converges absolutely, conditionally, or not at all.

Solution : Notice that this series is not positive nor is it alternating since the first few terms are approximately

$$\sum_{n=1}^{\infty} \frac{\cos n}{n^2} \approx \frac{0.540}{1^2} - \frac{0.416}{2^2} - \frac{0.990}{3^2} + \frac{0.284}{4^2} + \cdots .$$

First we check absolute convergence. $\sum_{n=1}^{\infty} \left|\frac{\cos n}{n^2}\right|$ looks a lot like the p-series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ with p = 2 > 1. We can use the direct comparison test. Since $0 \le |\cos n| \le 1$,

$$0 \le \left|\frac{\cos n}{n^2}\right| \le \frac{1}{n^2}$$

for all n. Since the p-series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, so does $\sum_{n=1}^{\infty} \left|\frac{\cos n}{n^2}\right|$ by the direct comparison test (Theorem). So the series of absolute values converges. The original series is absolutely convergent. We need to check further.

Conditional Convergence

A series Σu_n is said to be **conditionally convergent,** if

(i) Σu_n is convergent and

(ii) Σu_n is not absolutely convergent.

Example : The series $\Sigma u_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$ is conditionally convergent, since the given series is convergent (by Leibnitz Test) and $\sum |u_n| = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$ is not convergent, i.e., Σu_n is not absolutely convergent.

Example : Determine whether $\sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n^2-1}}$ converges absolutely, conditionally, or not at all.

Solution : First we check absolute convergence. $\sum_{n=2}^{\infty} \left|\frac{(-1)^n}{\sqrt{n^2-1}}\right| = \sum_{n=2}^{\infty} \frac{1}{\sqrt{n^2-1}}$.

Notice that $\frac{1}{\sqrt{n^2-1}} \approx \frac{1}{n}$. So let's use the limit comparison test. The terms of the series are positive and

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{\sqrt{n^2-1}} \cdot \frac{n}{1} = \lim_{n\to\infty} \frac{n}{\sqrt{n^2}} = \lim_{n\to\infty} \frac{n}{n} = 1 > 0.$$

Since the harmonic series $\sum_{n=2}^{\infty}\frac{1}{n}$ diverges (p-series with p = 1), then $\sum_{n=2}^{\infty}\left|\frac{(-1)^n}{\sqrt{n^2-1}}\right|$ diverges by the limit comparison test. So the series does not converge absolutely.

Since the series is alternating and not absolutely convergent, we check for conditional convergence using the alternating series test with $a_n = \frac{1}{\sqrt{n^2-1}}$. Check the two conditions.

1. $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{1}{\sqrt{n^2-1}} = 0.$

2. Further $a_{n+1} \le a_n$ is decreasing because $\frac{1}{\sqrt{(n+1)^2-1}} < \frac{1}{\sqrt{n^2-1}}$. (You could also show the derivative is negative.) Since the two conditions of the alternating series test are satisfied, $\sum_{n=2}^{\infty}\frac{(-1)^n}{\sqrt{n^2-1}}$ is conditionally convergent by the alternating series test.

Example : Determine whether $\sum_{n=1}^{\infty}\left|\frac{(-1)^n}{\sqrt[3]{n^2}}\right| = \sum_{n=1}^{\infty}\frac{1}{n^{2/3}}$ is a p-series with p = $\frac{2}{3} \le 1$. So the series of absolute values diverges. The original series is not absolutely convergent.

Since the series is alternating and not absolutely convergent, we check for conditional convergence using the alternating series test with $a_n = \frac{1}{n^{2/3}}$. Check the two conditions.

1. $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{1}{n^{2/3}} = 0.$

2. Further $a_{n+1} \le a_n$ because $\frac{1}{(n+1)^{2/3}} < \frac{1}{n^{2/3}}$.

Since the two conditions of the alternating series test are satisfied, $\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt[3]{n^2}}$ is conditionally convergent by the alternating series test..

Theorem : Every absolutely convergent series is convergent. The converse need not be true.

Proof : Let $\Sigma\, u_n$ be an absolutely convergent series.

Then $\Sigma\, |u_n|$ is convergent.

By cauchy's General Principle of Convergence, for any $\varepsilon > 0$, there exists a positive integer m s.t.

$$\big|\,|u_{m+1}| + |u_{m+2}| + \ldots + |u_n|\,\big| < \varepsilon, \quad \forall\, n \ge m$$

$\Rightarrow \quad |u_{m+1}| + |u_{m+2}| + \ldots + |u_n| < \varepsilon, \quad \forall\, n \ge m.$

Now $\quad |u_{m+1} + u_{m+2} + \ldots + u_n| \le |u_{m+1}| + |u_{m+2}| + \ldots + |u_n| < \varepsilon\ \forall\, n \ge m.$

Hence $\Sigma\, u_n$ is convergent (by Cauchy's Principle of Convergence).

(ii) **The converse of the theorem is not true** i.e.,

A convergent series may not be absolutely convergent.

Consider the series

$$\sum u_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$$

We have seen that the given series is convergent, by Leibnitz's Test but $\Sigma\,|u_n| = 1 + \frac{1}{2} + \frac{1}{3} + \ldots = \Sigma\,\frac{1}{n}$ is not convergent i.e. $\Sigma\, u_n$ is not absolutely convergent.

Example : Determine whether $\sum_{n=1}^{\infty}\frac{(-1)^n(2n^4+7)}{6n^9-2n}$ converges absolutely, conditionally, or not at all.

Solution. First we check absolute convergence. $\sum_{n=1}^{\infty}\left|\frac{(-1)^n(2n^4+7)}{6n^9-2n}\right| = \sum_{n=1}^{\infty}\frac{2n^4+7}{6n^9-2n}$.

Notice that $\frac{2n^4+7}{6n^9-2n} \approx \frac{1}{n^5}$. So let's use the limit comparison test. The terms of the series are positive and

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{2n^4+7}{6n^9-2n}\cdot\frac{n^5}{1} = \lim_{n\to\infty}\frac{2n^9+7n^5}{6n^9-2n} = \lim_{n\to\infty}\frac{2n^9}{6n^9} = \frac{1}{3} > 0.$$

Since $\sum_{n=1}^{\infty}\frac{1}{n^5}$ converges (p-series with p = 5 > 1), then $\sum_{n=1}^{\infty}\left|\frac{(-1)^n(2n^4+7)}{6n^9-2n}\right|$ converges by the limit comparison test. So the series converges absolutely.

Example : Determine whether $\sum_{n=2}^{\infty}\frac{(-1)^n}{\ln n}$ converges absolutely, conditionally, or not at all.

Solution : First we check absolute convergence. $\sum_{n=2}^{\infty}\left|\frac{(-1)^n}{\ln n}\right| = \sum_{n=2}^{\infty}\frac{1}{\ln n}$. We use the direct comparison test with $\frac{1}{n\ln n}$. Notice that $0 < \frac{1}{n\ln n} \le \frac{1}{\ln n}$ because n > 1. Next $\sum_{n=2}^{\infty}\frac{1}{n\ln n}$ diverges (To check that $\sum_{n=2}^{\infty}\frac{1}{n\ln n}$ diverges, use the integral test and u-substitution with u = ln x. $\int_2^{\infty}\frac{1}{x\ln x}dx = \lim_{b\to\infty}\ln|\ln x|_2^b = \lim_{n\to\infty}$ ln |ln b| – ln(ln 2) = +∞. Consequently $\sum_{n=2}^{\infty}\left|\frac{1}{\ln n}\right|$ diverges by the direct comparison test. So the series does not converges absolutely.

Since the series is alternating and not absolutely convergent, we check for conditional convergence using the alternating series test with $a_n = \frac{1}{\ln n}$. Check the two conditions.

1. $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{1}{\ln n} = 0.$

2. Further a_n is decreasing since $f(x) = \frac{1}{\ln n} = (\ln x)^{-1}$ then $f'(x) = -\frac{(\ln x)^{-2}}{x} <$ for $x \ge 2$.

Since the two conditions of the alternating series test are satisfied, $\sum_{n=2}^{\infty}\frac{(-1)^n}{\ln n}$ is conditionally convergent by the alternating series test.

Example : Determine whether $\sum_{n=2}^{\infty}\frac{(-1)^n 2^n}{2n}$ converges absolutely, conditionally, or not at all.

Solution : First we check absolute convergence. $\sum_{n=1}^{\infty}\left|\frac{2^n}{2n}\right| = \sum_{n=2}^{\infty}\frac{2^n}{2n}$. Use the nth term test:

$$\lim_{n\to\infty}\frac{2^n}{2n} = \lim_{n\to\infty}\frac{2^x}{2x} \overset{\text{L'Ho}}{=} \lim_{n\to\infty}\frac{2^x \ln 2}{2} = \infty.$$

Since $\lim_{n\to\infty} a_n \neq 0$ the series automatically diverges and cannot converge absolutely or conditionally.

When we test for absolute convergence using the ratio test, we can say more. If the ratio r is actually greater than 1, the series will diverge. We don't even need to check conditional convergence.

Theorem (The Ratio Test Extension) :

Assume that $\sum_{n=1}^{\infty} a_n$ is a series with non-zero terms and let $r = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$.

1. If $r < 1$, then the series $\sum_{n=1}^{\infty} a_n$ converges absolutely.
2. If $r > 1$ (including ∞), then the series $\sum_{n=1}^{\infty} a_n$ diverges.
3. If $r = 1$, then the test is inconclusive. The series may converge or diverge.

This is most helpful when the series diverges. It says we can check for absolute convergence and if we find the absolute value series diverges, then the original series diverges. We don't have to check for conditional convergence.

Example : Determine whether $\sum_{n=1}^{\infty}\frac{(-1)^n n!}{3^n}$ converges absolutely, conditionally, or not at all.

Solution : Here's a perfect place to use the ratio test because there is a factorial.

$$r = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}(n+1)!}{3^{n+1}}\cdot\frac{3^n}{(-1)^n n!}\right| = \lim_{n\to\infty}\left|\frac{n+1}{3}\right| = \infty.$$

The (original) series diverges by the ratio test. The ratio test says we don't have to check for conditional convergence.

Example : Determine whether $\sum_{n=1}^{\infty}\frac{(-1)^n(n+1)!}{2^n n}$ converges absolutely, conditionally, or not at all.

Solution : First we check absolute convergence using the ratio test because of the factorial.

$$r = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}(n+2)!}{2^{n+1}(n+1)}\cdot\frac{2^n n}{(-1)^n(n+1)!}\right| = \lim_{n\to\infty}\left|\frac{(n+2)n}{2(n+1)}\right| = \lim_{n\to\infty}\left|\frac{n^2+2n}{2n+2}\right| \overset{\text{HPwrs}}{=} \lim_{n\to\infty}\frac{n}{2} = \infty.$$

The (original) series diverges by the ratio test extension.

Example : Determine whether $\sum_{n=1}^{\infty}\frac{n(-2)^n}{3^{n+1}}$ converges absolutely, conditionally, or not at all.

Solution : Check absolute convergence using the ratio test extension.

$$r = \lim_{n\to\infty}\left|\frac{(n+1)(-2)^{n+1}}{3^{n+2}}\cdot\frac{3^{n+1}}{n(-2)^n}\right| = \lim_{n\to\infty}\left|\frac{2(n+1)}{3n}\right| = \frac{2}{3} < 1.$$

The (original) series converges absolutely by the ratio test extension.

Example : Does the following series converge absolutely, converge conditionally, or diverge?

$$\sum_{n=0}^{\infty}\left(-\frac{4}{5}\right)^n$$

Solution. Let us look at the positive term series for this given series.

$$\sum_{n=0}^{\infty}\left(\frac{4}{5}\right)^n$$

This is a geometric series with ratio, r = 4/5, which is less than 1.

Therefore this series converges, and the given series converges absolutely.

Example : Does the following series converge absolutely, converge conditionally, or diverge?

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$$

Solution. Let us look at the positive term series for the given series.

$$\sum_{n=1}^{\infty}\frac{1}{n}$$

This is the harmonic series and it diverges, so the given series will not converge absolutely.

Now we must determine if the given series will converge conditionally or diverge. To do this, we will have to look at the alternating series. To do this, we must use the alternating series test.

$$\text{Let } u_n = \frac{1}{n}.$$

$u_n > 0$ for all $n \geq 1$, so the first condition of this test is satisfied.

Now we must determine if the second condition is satisfied. This is easy to see. As n gets larger, the fraction 1/n gets smaller. So $u_n \geq u_{n+1}$ and the second condition is true. Now let us determine if the third condition is satisfied.

$$\lim_{n\to\infty}\frac{1}{n} = 0$$

The third condition holds, so the alternating series converges, and the given series converges conditionally.

So here the steps you will need to follow when determining absolute convergence, conditional convergence or divergence of a series. Look at the positive term series first. If the positive term

A. If it converges, then the given series converges absolutely.

B. If the positive terms series diverges, use the alternating series test to determine if the alternating series converges. If this alternating series diverges, then the given series diverges.

Example : Does the following series converge absolutely, converges conditionally, or diverges?

$$\sum_{n=1}^{\infty}\frac{(-4)^n}{n^2}$$

Solution. Here is the positive term series.

$$\sum_{n=1}^{\infty}\frac{(4)^n}{n^2}$$

we are going to use the ratio test to determine the convergence of this series.

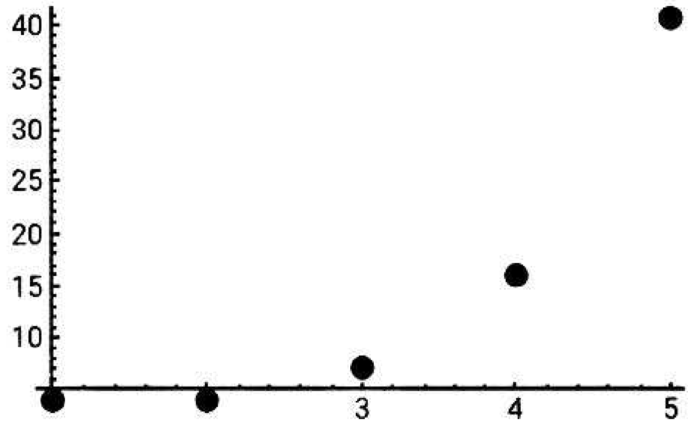
$$\lim_{n\to\infty}\frac{\frac{4^{n+1}}{(n+1)^2}}{\frac{4^n}{n^2}} = \lim_{n\to\infty}\frac{4^{n+1}}{(n+1)^2}\cdot\frac{n^2}{4^n} = \lim_{n\to\infty}\frac{4n^2}{(n+1)^2} = \lim_{n\to\infty}\frac{8n}{2n+2}$$

$$= \lim_{n \to \infty} \frac{8}{2} = 4 = \rho > 1$$

So the positive term series diverges by the ratio test, and the given series does not converge absolutely. Therefore, we will have to look at the alternating series to determine if it converges or not.

Let $u_n = \frac{4^n}{n^2}$.

u_n is positive for $n \geq 1$, so the first condition is satisfied. Now to determine it the second condition is holds. To help me determine this, we will plot the first 5 term of this sequence.



Therefore, the third condition is not satisfied because the terms of this sequence are increasing. In fact 4^n grows faster than n^2. So the alternating series diverges, and the given series also diverges.

Example : Does the following series converge absolutely, converge conditionally, or diverge?

$$\sum_{n=1}^{\infty} \frac{\cos n\pi}{n}$$

Solution. Since the cos $n\pi$ is the alternating term, the positive term series is the harmonic series. Remember that the harmonic series diverges, so the given series does not converge absolutely. Now to determine the convergence of the alternating series.

Let $u_n = \frac{1}{n}$.

$u_n > 0$ for all $n \geq 1$, so the first condition of this test satisfied.

Now we must determine if the second condition is satisfied. This is easy to see. As n gets larger, the fraction 1/n gets smaller. So $u_n \geq u_{n+1}$ and the second condition is true. Now let us determine if the third condition is satisfied.

$$\lim_{n \to \infty} \frac{1}{n} = 0$$

The third condition holds, so the alternating series converges, and the given series converges conditionally.

Example : Does the following series converge absolutely, converge conditionally, or diverge ?

$$\sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{n+2}{3n-1}\right)^n$$

Solution. Here is the positive term series.

$$\sum_{n=1}^{\infty} \left(\frac{n+2}{3n-1}\right)^n$$

Since nth term is raised to the nth power, we will use the nth-root test to determine convergence or divergence of this series.

$$a_n = \left(\frac{n+2}{3n-1}\right)^n \rightarrow \sqrt[n]{a_n} = \frac{n+2}{3n-1}$$

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \frac{1}{3} = \frac{1}{3} = \rho < 1$$

The positive term series converges by the nth-root test. Therefore, the given series converges absolutely.

6. CONVERGENCE OF THE INFINITE INTEGRAL $\int_1^\infty u(x)\,dx$

Definition : The infinite integral $\int_1^\infty f(x)\,dx$ is said to be **convergent (divergent)**, if $\lim_{t\to\infty}\int_1^t f(x)dx$ is finite (infinite).

We state a result without proof :

Theorem : If $f(x) \geq 0 \ \forall\ x \geq 1$, then $\int_1^\infty f(x)\,dx$ is convergent iff there exists a positive number k such that for all $t \geq 1$, $\int_1^t f(x)\,dx \leq k$.

Example : Examine the convergence and divergence of

(i) $\int_1^\infty \frac{dx}{x}$, (ii) $\int_1^\infty \frac{dx}{1+x^2}$.

Solution :

(i) We have

$$\int_1^t \frac{dx}{x} = |\log x|_1^t = \log t.$$

$$\therefore \quad \lim_{t\to\infty}\int_1^t \frac{dx}{x} = \lim_{n\to\infty}(\log t) = \infty. \quad \text{Here } \int_1^\infty \frac{dx}{x} \text{ diverges.}$$

(ii) Now $\int_1^t \frac{dx}{1+x^2} = |\tan^{-1}x|_1^t = \tan^{-1}t - \frac{\pi}{4}$.

$$\therefore \quad \lim_{n\to\infty}\int_1^t \frac{dx}{1+x^2} = \lim_{n\to\infty}\left(\tan^{-1}t - \frac{\pi}{4}\right) = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}.$$

Hence $\int_1^\infty \frac{dx}{1+x^2}$ converges.

Definition : Let f(x) be a real valued function with domain $[1, \infty[$.

The function f(x) is said to be **non-negative,** if

$$f(x) \geq 0 \ \forall\ x \geq 1.$$

The function f(x) is said to be **monotonically decreasing,** if

$$x \leq y \Rightarrow f(x) \geq f(y)\ ;\ x, y \in [1, \infty[.$$

For example, $f(x) = 1/x^2$ is non-negative and monotonically decreasing $\forall\ x \geq 1$.

Cauchy's Integral Test

If u(x) is a non-negative, monotonically decreasing and integrable function such that $u(n) = u_n\ \forall\ n \in N$, then the series $\sum_{n=1}^\infty u_n$ is convergent if and only if $\int_1^\infty u(x)\,dx$ is convergent.

Proof : Since u(x) is monotonically decreasing, so

$$u(n) \geq u(x) \geq u(n+1), \text{ when } n \leq x \leq n+1.$$

Since u(x) is non-negative and integrable, so

$$\int_n^{n-1} u(n)dx \geq \int_n^{n+1} u(n)dx \geq \int_n^{n+1} u(n+1)dx$$

or $\quad u(n)(n+1-n) \geq \int_n^{n-1} u(n)\, dx \geq u(n+1)\,(n+1-1)$

$\therefore \quad u_n \geq \int_n^{n+1} u(x)\, dx \geq u_{n+1}.\ \ (\because u(n) = u_n).$ ...(1)

Putting n = 1, 2, ..., n – 1 in (1) and adding, we get

$$(u_1 + u_2 + \ldots + u_{n-1}) \geq \left[\int_1^2 + \int_2^3 + \ldots \int_{n-1}^n \right] \geq (u_2 + u_3 + \ldots + u_n)$$

or $\quad S_n - u_n \ {}^3 \int_1^n u(x)dx \geq S_n - u_1,$ where $S_n = u_1 + u_2 + \ldots + u_{n-1} + u_n$

or $\quad S_n - u_1 \leq \int_1^n u(x)\, dx \leq S_n - u_n \leq S_n$, as $u_n \geq 0$ ...(2)

The condition is necessary.

Suppose the series $\sum_{n=1}^{\infty} u_n$ is convergent.

Then there exists a positive number K such that

$S_n \leq K \ \ \forall \ \ n.$ ...(3)

From (2) and (3), $\int_1^n u(x)\, dx \leq S_n \leq K \ \ \forall\, n$

Hence $\int_1^{\infty} u(x)dx$ is convergent.

The condition is sufficient.

Suppose $\int_1^{\infty} u(x)\, dx$ is convergent.

Then there exists a positive number k such that

$\int_1^n u(x)dx \leq k \ \ \forall \ n.$ ...(4)

From (2) and (4), $\quad S_n - u_1 \leq \int_1^n u(x)dx \leq k \ \ \forall\ n$

or $\quad S_n \leq k + u_1\ \forall$ n, so that $\langle S_n \rangle$ is bounded above.

Hence the series $\Sigma\, u_n$ is convergent.

Example : Show that the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ (p > 0) is convergent if p > 1 and divergent if p ≤ 1.

Solution : Let $u_n = \frac{1}{n^p}$ and $u(x) = \frac{1}{x^p}$, so that $u(n) = u_n \ \forall \ n \in N$.

Clearly, for x ≥ 1, u(x) is non-negative, integrable and a decreasing function of x. Now

$$\int_1^t u(x)dx = \int_1^t x^{-p}dx = \frac{1}{(-p+1)}\left|x^{-p+1}\right|_1^t, \text{ if } p \neq 1,$$

$$= \left|\log x\right|_1^t, \text{ if } p = 1.$$

$$\therefore \quad \int_1^t u(x)\, dx = \frac{1}{1-p}(t^{1-p} - 1), \text{ if } p \neq 1,$$

$$= (\log t - \log 1), \text{ if } p = 1.$$

$$\text{Now} \quad \lim_{n\to\infty} \int_1^t u(x)dx = \begin{cases} \frac{1}{1-p}\left[\lim_{n\to\infty} t^{1-p} - 1\right], & \text{if } p \neq 1 \\ \lim_{n\to\infty} \log t \quad , & \text{if } p = 1 \end{cases} \qquad (\because \log 1 = 0)$$

$$= \begin{cases} \infty \quad , & \text{if } p < 1, \\ \frac{1}{1-p}(0-1) = \frac{1}{p-1}, & \text{if } p > 1, \\ \infty \quad , & \text{if } p = 1. \end{cases}$$

$$\text{So} \quad \lim_{n\to\infty} \int_1^t u(x) = \infty, \text{ if } p \leq 1 \text{ and } \frac{1}{p-1}, \text{ if } p > 1$$

If Follows that the improper integral $\int_1^{\infty} u(x)$ is convergent if p > 1 and divergent if p ≤ 1. Hence, by Cauchy's integral test, $\sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if p > 1 and divergent if p ≤ 1.

Example : Show that the series $\sum_{n=2}^{\infty} \frac{1}{n\,(\log n)^p}$ is convergent if p > 1 and divergent if 0 < p ≤ 1.

Solution : Let $u(x) = \frac{1}{x(\log x)^p}$, (p > 0)

Then for x ≥ 2, u(x) is non-negative, monotonically decreasing and integrable function. Also $u(n) = u_n \ \forall \ n \in N$.

$$\text{Consider} \int_2^t u(x)dx = \int_2^t \frac{dx}{x(\log x)^p}$$

$$= \frac{1}{-p+1}\left|(\log x)^{-p+1}\right|_2^t, \text{ if } p \neq 1$$

$$= \left|\log(\log x)\right|_2^t, \text{ if } p = 1$$

or $$\int_2^t u(x) = dx = \begin{cases} \frac{1}{1-p}[(\log t)^{1-p} - (\log 2)^{1-p}], & \text{if } p \neq 1, \\ [\log(\log t) - \log(\log 2)], & \text{if } p = 1. \end{cases}$$

$$\therefore \quad \lim_{n\to\infty} \int_2^t u(x)dx = \begin{cases} \infty & , \text{ if } p < 1, \\ \frac{1}{p-1}(\log 2)^{1-p}, & \text{if } p > 1, \\ \infty & , \text{ if } p = 1. \end{cases}$$

Thus $\int_2^\infty u(x)\,dx$ is convergent if $p > 1$ and divergent if $0 < p \leq 1$. Hence, by Cauchy's Integral Test, the given series is convergent if $p > 1$ and divergent if $0 < p \leq 1$.

Example : Test for convergence the series $\sum_{n=2}^{\infty} \frac{1}{n \log n}$.

Solution : consider the case $p = 1$.

We know $\lim_{n\to\infty} \int_2^t u(x)\,dx = \lim_{n\to\infty} [\log(\log t) - \log(\log 2)] = \infty$.

Hence the given series diverges.

Example : Apply Cauchy's Integral Test to examine the convergence of the following series :

(i) $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$, (ii) $\sum_{n=1}^{\infty} \frac{1}{n^2+n}$.

Solution : (i) Let $u(x) = \frac{1}{x^2+1}$, so that $u(n) = u_n \ \forall \ n \in N$.

For $x \geq 1$, $u(x)$ is non-negative, monotonically decreasing and integrable function. Now

$$\lim_{t\to\infty} \int_1^t \frac{dx}{x^2+1} = \lim_{t\to\infty} \left|\tan^{-1} x\right|_1^{\infty} = \lim_{n\to\infty} \left(\tan^{-1} t - \frac{\pi}{4}\right)$$

$$= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}, \text{ which is finite.}$$

Thus $\int_{n=1}^{\infty} \frac{dx}{x^2+1}$ is convergent and so $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ is convergent.

(By Cauchy's Integral Test)

(ii) Let $u(x) = \frac{1}{x^2+x}$, so that $u(n) = u_n \ \forall \ n \in N$.

For $x \geq 1$, $u(x)$ is non-negative, monotonically decreasing and integrable function. Now

$$\int_1^t \frac{dx}{x^2+x} = \int_1^t \frac{dx}{x(x+1)} = \int_1^t \left(\frac{1}{x} - \frac{1}{x+1}\right)dx$$

$$= \left|\log x - \log(x+1)\right|_1^t = \left|\log\frac{x}{1+x}\right|_1^t$$

$$= \log\frac{t}{1+t} - \log\frac{1}{2} = \log\frac{1}{1+(1/1)} - \log\frac{1}{2}.$$

$\therefore$ $\lim\limits_{n\to\infty}\int_1^t \frac{dx}{x^2+x}$ = log 1 + log 2 = log 2, which is finite.

Thus $\int_1^\infty u(x)\,dx$ is a convergent and so by Cauchy's Integral Test, the given series is convergent.

Example : Test the convergent of $\sum\limits_{n=3}^{\infty}\frac{1}{n\log n(\log\log n)^p}$, $p > 0$.

Solution : Let $u(x) = \frac{1}{x\log x(\log\log x)^p}$, so that $u(n) = u_n\ \forall\ n$.

For $x \geq 3$, u(x) is non-negative, monotonically decreasing and integrable function. Now

$$\int_3^t u(x)dx = \int_{y_1}^{y_2}\frac{dy}{y^p},$$

where $y = \log\log x$,

$y_1 = \log\log 3$, $y_2 = \log\log t$

$$= \frac{1}{1-p}\left|y^{1-p}\right|_{y_1}^{y_2},\ \text{if } p \neq 1$$

$$= \left|\log y\right|_{y_1}^{y_2},\quad \text{if } p = 1$$

or $$\int_3^t u(x)dx = \begin{cases}\frac{1}{1-p}[(\log\log t)^{1-p} - (\log\log 3)^{1-p}], & \text{if } p \neq 1,\\ \log(\log\log t) - \log(\log\log 3)], & \text{if } p = 1.\end{cases}$$

$$\therefore\quad \lim_{t\to\infty}\int_3^t u(x)dx = \begin{cases}\infty & , \text{ if } p < 1,\\ \frac{1}{p-1}(\log\log 3)^{1-p}, & \text{if } p > 1,\\ \infty & , \text{ if } p = 1.\end{cases}$$

Thus $\int_3^t u(x)\,dx$ is convergent if $p > 1$ and divergent if $0 < p \leq 1$.

Hence, by Cauchy's Integral Test, the given series is convergent if $p > 1$ and divergent if $0 < p \leq 1$.

Unit-3 | Ponit Set Topology

1. SOME BASIC DEFINITIONS

Sets

A set is a well defined collection of objects. The objects of the set are called elements or members of set of Capital letters X, Y, S, T etc. generally denote sets, while small letters x, y, s, t etc. denote elements of a set. If x is an element of set S, then we write $x \in S$ and read it as 'x belongs to S' or 'x is a member of S'. If x is not an element of a set S, then we write $x \notin S$ and read it as 'x does not belong to S' or 'x is not a member of S'.

A set having no element is called an empty set or a null set. It is denoted by ϕ or { }.

A non-empty set S, written as $S \neq \phi$, is a set which has at least one element. S = {1}

A set is said to be finite or infinite according to the number of elements present in the set.

Examples

1. $N = \{1, 2, 3, 4, ...\}$ is the set of natural numbers.
   Here $0 \notin N$.
2. $I = \{ ..., -3, -2, -1, 0, 1, 2, 3, ...\}$ is the set of integers.
3. The set $S = \{11, 12, 13, 14, ...\}$ may be written as $S = \{x : x \in N \text{ and } x > 10\}$.
4. The set $T = \{..., -15, -10, -5, 0, 5, 10, 15, ...\}$ may be written as $T = \{5x : x \in I\}$.
5. The set $S = \{x : x \in N \text{ and } 2 < x < 3\}$ is an empty set i.e., $S = \phi$.

Set Theoretic Concepts

Definition 1 : If S and T be two sets such that every element of S is also an element of T, then we say that S is a **subset** of T or S is **contained** in T.

Symbolically, $S \subset T$, if $x \in S \Rightarrow x \in T \ \forall \ x \in S$.

If S is contained in T, then we also say that T contains S or T is a **super set** of S and write it as $T \supset S$.

Thus $S \subset T \Leftrightarrow T \supset S$.

Example

1. Let $S = \{1, 3, 5\}$ and $T = \{1, 2, 3, 4, 5\}$. Then $S \subset T$.
2. Let $S = \{1, 3, 6\}$ and $T = \{1, 2, 3, 4, 5\}$.
   Then $S \not\subset T$, since $6 \in S$ but $6 \notin T$.

Note :- The empty set ϕ is a subset of every set,

Definition 2 : Two sets S and T are said to be **equal**, written as S = T, if and only if $S \subset T$ and $T \subset S$. Or we can say that two sets are said to be **equal** if they contain the same elements.

Example

$\{2, 3, 5, 7\} = \{7, 2, 5, 3\}$ and $\{2, 3, 5, 7\} \neq \{2, 5, 7, 6\}$.

Definition 3 : The **union** of two sets S and T, denoted as $S \cup T$, is defined as

$$S \cup T = \{x : x \in S \text{ or } x \in T\}.$$

In other words, $S \cup T$ contains all the elements of S as well as T.

Definition 4 : The **intersection** of two sets S and T, denoted as $S \cap T$, is defined as

$$S \cap T = \{x : x \in S \text{ and } x \in T\}.$$

In other words, $S \cap T$ contains the common elements of S and T.

Definition 5 : Two sets S and T are said to be **disjoint** if $S \cap T = \phi$.

Examples

1. Let $S = \{1, 3, 5\}$, $T = \{2, 3, 4, 5, 7\}$
   Then $S \cup T = \{1, 2, 3, 4, 5, 7\}$ and $S \cap T = \{3, 5\}$.
2. Let $E = \{..., -6, -4, -2, 0, 2, 4, 6, ...\}$,
   $O = \{... -5, -3, -1, 1, 3, 5, ...\}$.
   Then $E \cup O = I$ (set of all integers)
   and $E \cap O = \phi$. Thus E and O are disjoint sets.

Note :

$$\boxed{\begin{array}{l} x \in S \cup T \Rightarrow x \in S \text{ or } x \in T. \\ x \in S \cap T \Rightarrow x \in S \text{ and } x \in T. \\ x \notin S \cup T \Rightarrow x \notin S \text{ or } x \notin T. \\ x \notin S \cap T \Rightarrow x \notin S \text{ and } x \notin T. \end{array}}$$

Definition 6. A non-empty set Δ is said to be an index set for the family $S = \{S_\lambda\}$ of sets if for each $\lambda \in \Delta$, there exists a set S_λ in the family S.

For Example, $N = \{1, 2, 3, 4, ..., ..\}$ is an index set for the family of sets $\{S_1, S_2, S_3, S_4, ...\}$.

Consider an arbitrary family of sets $\{S_\lambda : \lambda \in \Delta\}$, Δ is any index set.

Definition 7. The union of the sets $\{S_\lambda\}$ is defined as

$$\bigcup_{\lambda \in \Delta} S_\lambda = \{x : x \in S_\lambda \text{ for some } \lambda \in \Delta\}.$$

Definition 8. The intersection of the sets $\{S_\lambda\}$ is defined as

$$\bigcap_{\lambda \in \Delta} S_\lambda \{x : x \in S_\lambda \text{ for each } \lambda \in \Delta\}.$$

Definition 9. If S and T are two sets, then the difference of S and T, denoted as $S \sim T$, is defined as

$$S \sim T = \{x : x \in S \text{ and } x \notin T\}.$$

For Example, if $S = \{1, 2, 3, 4, 8\}$ and $T = \{2, 3, 5, 6, 7\}$, then
$S \sim T = \{1, 4, 8\}$.

Definition 10. If S is a subset of a set X, then the complement of S, denoted by $X \sim S$, is defined as

$$X \sim S = \{x : x \in X \text{ and } x \notin S\}.$$

Other notations of the complement of S and S' or S^c.

Thus $x \in X \sim S \Rightarrow x \in X$ but $x \notin S$.

For Example, if $X = \{1, 2, 3, 4, ..., 10\}$, and $S = \{2, 4, 6, 8, 10\}$, then $X \sim S = \{1, 3, 5, 7, 6\}$.

Remarks

- $S \cup S = S$, $S \cap S = S$.
- $S \cup T = T \cup S$, $S \cap T = T \cap S$.
- $S \cup \phi = S$, $S \cap \phi = \phi$.
- $S_\lambda \subset \bigcup_{\lambda \in \Delta} S_\lambda$ for each $\lambda \in \Delta$.

- $\bigcap_{\lambda \in \Delta} S_\lambda \subset S_\lambda$ for each $\lambda \in \Delta$.
- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.
- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.
- (De Morgan's Laws)

  If S and T be any two subsets of a set X, then

  (i) $X \sim (S \cup T) = (X \sim S) \cap (X \sim T)$.

  (ii) $X \sim (S \cap T) = (X \sim S) \cup (X \sim T)$.

Proof. We recall that

$A = B \Leftrightarrow A \subset B$ and $B \subset A$.

(i) We see that $x \in X \sim (S \cup T)$

$\Leftrightarrow x \in X$ but $x \notin S \cup T$

$\Leftrightarrow x \in X$ but $x \notin S$ and $x \notin T$

$\Leftrightarrow x \in X \sim S$ and $x \in X \sim T$

$\Leftrightarrow x \in (X \sim S) \cap (X \sim T)$.

Hence $X \sim (S \cup T) = (X \sim S) \cap (X \sim T)$.

(ii) Now $x \in X \sim (S \cap T)$

$\Leftrightarrow x \in X$ but $x \notin S \cup T$

$\Leftrightarrow x \in X$ but $x \notin S$ or $x \notin T$

$\Leftrightarrow x \in X \sim S$ or $x \in X \sim T$

$\Leftrightarrow x \in (X \sim S) \cup (X \sim T)$.

Hence $X \sim (S \cap T) = (X \sim S) \cup (X \sim T)$.

- The above laws can be generalised as follow :

(i) $X \sim \left(\bigcup_{\lambda \in \Delta} S_\lambda \right) = \bigcup_{\lambda \in \Delta} (X \sim S_\lambda)$.

(ii) $X \sim \left(\bigcap_{\lambda \in \Delta} S_\lambda \right) = \bigcup_{\lambda \in \Delta} (X \sim S_\lambda)$.

(Here each S_λ is a subset of X).

Function or Mapping

Definition 1 : If S and T are two non-empty sets, then a **function** from S to T is a rule, denoted as

$$F : S \to T$$

which associates to each element $x \in S$ a unique element $y \in T$.

The element y, written as $y = f(x)$, is called the **image** of x and x is called a **pre-image** of y. The set S is called the **domain** and T the **co-domain** of the function $f : S \to T$.

The set $f(S) = \{f(x) : x \in S\}$ is called the **range** of the function $f : S \to T$.

$f : N \to N$ defined by $f(n) = n^2$ is a function whose range is $\{1, 4, 9, 16, 25, ...\}$.

Definition 2 : A mapping $f : S \to T$ is called **one-to-one**, if

$$x \neq y \Rightarrow f(x) \neq f(y) \ \forall \ x, y \in S.$$

Or

$$f(x) = f(y) \Rightarrow x = y \quad \forall \ x, y \in S.$$

Definition 3 : A mapping $f : S \to T$ is called **onto,** if for each $t \in T$, there exists some $s \in S$ such that $f(s) = t$.

Definition 4 : Two mappings $f : S \to T$ and $g : S \to T$ are said to be **equal**, if

$$f(x) = g(x) \ \forall \ x \in S.$$

Definition 5 : The **product** or the **composition** of two mappings

$$f : S \to T \quad \text{and} \quad g : T \to U$$

is defined as the mapping

$$gof : S \to U$$

such that $(gof)(x) = g(f(x)) \ \forall \ x \in S$.

Examples

1. $f : N \to N$ defined as $f(x) = x^2 \ \forall \ x \in N$ is one-to-one,

   since for any $x, y \in N$,

   $$x \neq y \Rightarrow x^2 \neq y^2 \Rightarrow f(x) \neq f(y).$$

   However, f is not onto ; since for $2 \in N$, there does not exist any $x \in N$ such that $f(x) = 2$. (Notice that $x^2 = 2 \Rightarrow x = \pm \sqrt{2} \notin N$)

2. $f : I \to I$ defined as $f(x) = x + 3 \ \forall \ x \in I$ is one-to-one and onto.

   Let $x, y \in I$ then $f(x) = f(y) \Rightarrow x + 3 = y + 3 \Rightarrow x = y$ so f is one-to-one. Let $x \in I$ be arbitrary. Then $y = x - 3 \in I$ and $f(y) = x - 3 + 3 = x$ and so f is onto.

3. $f : I \to I$ defined by $f(x) = x^2 \ \forall \ x \in I$, is not one-to-one.

   Notice that $2 \neq -2$ but $f(2) = f(-2) = 4$. Also f is not onto.

4. Let $f : N \to N$ and $g : N \to N$ be defined as

   $$f(x) = 2x + 3 \text{ and } g(x) = x^2 \ \forall \ x \in N.$$

   Then $gof : N \to N$ is given by

   $$(gof)(x) = g(f(x)) = g(2x + 3) = (2x + 3)^2.$$

   Also $fog : N \to N$ is given by

   $$(fog)(x) = f(g(x)) = f(x^2) = 2x^2 + 3.$$

   In general, $fog \neq gof$.

Real Numbers

We are familiar with the following sets of numbers :

$N = \{1, 2, 3, ...\}$ is the set of natural numbers.

$Z = \{......, -3, -2, -1, 0, 1, 2, 3, ...\}$ is the set of integers

The other notations of the set of integer are I or J.

$Q = \{p/q$: and q are integers and $q \neq 0\}$ is the set of rational numbers. Thus a rational numbers is of the form p/q, where p and q are integers and $q \neq 0$.

It is clear that $N \subset Z \subset Q$ (e.g., $3 = 3/1$, $0 = 0/1$, $-4 = -4/1$).

It may be noted that any rational number $x \in Q$ can be expressed as $x = p/q$, where p and q are co-prime integers i.e., the H.C.F. of p and q is 1, denoted as $(p, q) = 1$.

For Example, $6/8 = 3/4$, $9/15 = 3/5$ etc.

There exist numbers which are not expressible as p/q, where p and q are integers and $q \neq 0$, such numbers are called irrational numbers.

The set consisting of all rational and irrational numbers is called the set of real number, denoted by R. Thus $x \in R \Leftrightarrow x$ is either a rational number or x is an irrational number.

We have $N \subset Z \subset Q \subset R$.

Ex. Prove that $\sqrt{3}$ is not a rational number.

Sol. Let, if possible, $\sqrt{3}$ be a rational number.

Then $\sqrt{3}$ = p/q, where p and q are integers and $q \neq 0$.

We can further suppose that H.C.F. of p and q is 1.

Now $\sqrt{3} = p/q \Rightarrow p^2 = 3q^2$. ...(1)

Since $q \in I$, so $3q^2 \in I$ and 3 is a factor of $3q^2$.

By (1), 3 is a factor of $p^2 \Rightarrow$ 3 is a factor or p

$\Rightarrow$ $p = 3m$, for some $m \in I$.

Putting $p = 3m$ in (1), we get $q^2 = 3m^2$.

We have seen that $p^2 = 3q^2 \Rightarrow$ 3 is a factor of p.

Similarly, $q^2 = 3m^2 \Rightarrow$ 3 is a factor of q.

Thus, 3 is a common factor of p and q, which is a contradiction to the fact that H.C.F. of p and q is 1. Hence $\sqrt{3}$ must be an irrational number.

Ex. Prove that $\sqrt{8}$ is not a rational number.

Sol. Let, if possible, $\sqrt{8}$ be a rational number.

Then $\sqrt{8} = \frac{p}{q}$; where $p, q \in I$, $q > 0$.

We can further suppose that H.C.F. of p and q is 1.

Clearly $2 < \sqrt{8} < 3 \Rightarrow 2 < \frac{p}{q} < 3 \Rightarrow 2q < p < 3q \Rightarrow 0 < p - 2q < q$

$\Rightarrow$ $p - 2q$ is a positive integer less than q.

$\therefore$ $\sqrt{8}\,(p - 2q)$ is not an integer. ...(1)

Again $\sqrt{8}(p-2q) = \frac{p}{q}(p-2q) = \frac{p^2}{q} - 2p = \frac{p^2}{q^2}\cdot q - 2p = 8q - 2p$,

where $8q - 2p$ is an integer.

Thus $\sqrt{8}\,(p - 2q)$ is an integer. ...(2)

(1) and (2) are contradictory statements. Hence $\sqrt{8}$ is not a rational number.

Ex. Prove that there is no rational number whose cube is 2.

Sol. Let, if possible, $x^3 = 2$; $x \in Q$.

We may take x = p/q, where p and q are co-prime integers i.e., the H.C.F. of p and q is 1.

We have $x^3 = 2 \Rightarrow p^3 = 2q^3 \Rightarrow p^3$ is a multiple of 2 and so p must be a multiple of 2, otherwise p^3 will not a multiple of 2.

Let $p = 2r$, for some $r \in Z$. Putting $p = 2r$ in $p^3 = 2q^3$, we get $q^3 = 4r^3 \Rightarrow q^3$ is a multiple of 2 and so q must be a multiple of 2. Since p and q are both multiples of 2, it follows that 2 divides p and q, which contradicts the fact that p and q are co-prime. Hence our assumption is wrong and so there is no rational number whose cube is 2.

2. BOUNDED SET

Definition 1 : A real number K is called an **upper bound** of a set S, if every element of S is less than or equal to K

i.e. $x \leq K \quad \forall\, x \in S.$

If a set S has an upper bound, then the set S is called **bounded above.**

Definition 2 : A real number k is called a **lower bound** of a set S, if every element of S is greater than or equal to k.

i.e. $k \leq x \quad \forall\, x \in S.$

If a set S has a lower bound, then the set S is called **bounded below.**

Definition 3 : A set S is called **bounded** if it is both bounded above and bounded below.

or

A set S is called **bounded** if $\exists$ two real numbers k and K such that

$$k \leq x \leq K \quad \forall\, x \in S.$$

Examples

1. The set S = {1, 2, 3, ..., 10} is bounded, 1 being a lower bound and 10 being an upper bound. Indeed $1 \leq x \leq 10\ \forall\, x \in S$.
2. The set N = {1, 2, 3, ...} of natural numbers is bounded below but not bounded above, 1 being a lower bound. So N is not bounded.
3. The set S = {..., –3, –2, –1} is bounded above but not bounded below, –1 being an upper bound. So S is not bounded.
4. The set Z of integers is neither bounded above nor bounded below. So Z is not bounded.

   Similarly, the sets Q, R are not bounded.

Remarks

1. A set may or may not possess upper and/or lower bounds.

   e.g., N has a lower bound but has no upper bound.
2. Bounds of a set may or may not belong to the set.

   If $S = \left\{\frac{1}{n} : n \in N\right\}$, its lower bound $0 \notin S$ and upper bound $1 \in S$.
3. If a real number K is not an upper bound of a set S, then there exists at least one element $y \in S$ such that $y > K$.
4. If a real number k is not a lower bound of a set S, then there exists at least one element $z \in S$ such that $k > z$ i.e., $z < k$.
5. If a set has one upper bound, then it has an infinite number of upper bounds.

   Notice that if K is an upper bound of a set S, then any number K_1 (> K) is also an upper bound of S.
6. If a set has one lower bound, then it has an infinite number of lower bounds.

   Notice that if k is a lower bound of a set S, then any number k_1 (< k) is also a lower bound of S.

Remarks 5 and 6 lead to the following :

Definition 4 : The smallest of all the upper bounds of a set S is called the **supremum** of the set S or the **least upper bound** (l.u.b.) of the set S.

Definition 5 : The greatest of all the lower bounds of a set S is called the **infimum** of the set S or the **greatest lower bound** (g.l.b.) of the set S.

Notation : If M and m are the supremum and infimum of a set S respectively, we write

$$m = \inf S \quad \text{and} \quad M = \sup S.$$

m and M are called the **bounds** of the set S, provided they exist.

Remarks

1. A set may or may not have the supremum or/and infimum.
2. The supremum and infimum of a set may or not belong to the set.

Example

1. If $S = \{1, 2, 3, ..., 10\}$, then $\inf S = 1 \in S$, $\sup S = 10 \in S$.
2. If $S = \left\{1, \frac{1}{2}, \frac{1}{3}, ...\right\}$, then $\inf S = 0 \notin S$, $\sup S = 1 \in S$.
3. If $S = \left\{-1, -\frac{1}{2}, -\frac{1}{3}, ...\right\}$, $\inf S = -1 \in S$, $\sup S = 0 \notin S$.
4. If $S = \left\{-2, -\frac{3}{2}, -\frac{4}{3}, -\frac{5}{4}, ..., -\frac{n+1}{n}, ...\right\}$, then $\inf S = -2 \in S$ and $\sup S = -1 \notin S$.
5. If $S = \left\{1-1, 1+\frac{1}{2}, 1-\frac{1}{3}, 1+\frac{1}{4}, 1-\frac{1}{5}, 1+\frac{1}{6}, ..., 1+\frac{(-1)^n}{n}, ...\right\}$

   $= \left\{0, \frac{3}{2}, \frac{2}{3}, \frac{5}{4}, \frac{4}{5}, \frac{7}{6}, \frac{6}{7}, ...\right\}$, then $\inf S = 0 \in S$, $\sup S = \frac{3}{2} \in S$.
6. The set Z of integers has neither infimum nor supremum.

Ex. Give an example of a set S which is

(i) bounded above but not bounded below,

(ii) bounded below but not bounded above,

(iii) neither bounded above nor bounded below,

(iv) bounded.

Sol. (i) $S = \{..., -3, -2, -1\}$

is bounded above by -1, but not bounded below.

(ii) $N = \{1, 2, 3, ...\}$

is bounded below by 1, but not bounded above.

(iii) $I = \{..., -3, -2, -1, 0, 1, 2, 3, ...\}$

is neither bounded above nor bounded below.

(iv) $S = \left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ..., \frac{1}{n}, ...\right)$

is bounded, since $0 < \frac{1}{n} \le 1 \ \forall \ n \in N$.

Ex. Give an Example of a non-empty bounded subset S of **R** whose supremum and infimum both belong to **R** ~ S.

Sol. Let $S = \{x : 0 < x < 1\}$. Then $\sup S = 1 \in \mathbf{R} \sim S$ and $\inf S = 0 \in \mathbf{R} \sim S$.

Ex. Which of the following sets are bounded above, bounded below, bounded or otherwise:

(i) $A = \{3, 3^2, 3^3, ..., 3^n, ...\}$,

(ii) $B = \left\{2, \frac{3}{2}, \frac{4}{3}, ..., \frac{n+1}{n}, ...\right\}$,

(iii) $C = \left\{\frac{1}{n} : n \in N\right\}$,

(iv) $D = \{-1, 2, -3, 4 ..., (-1)^n n, ...\}$,

(v) $E = \left\{1 + \frac{(-1)^n}{n} : n \in N\right\}$,

(vi) $F = \left\{-1, -\frac{1}{2}, -\frac{1}{3}, -\frac{1}{4}, ...\right\}$,

(vii) $G = \left\{-2, -\frac{3}{2}, -\frac{4}{3}, ..., -\frac{n+1}{n}, ...\right\}$,

(viii) $S = \left\{-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, ... (-1)^n \cdot \frac{1}{n}, ...\right\}$,

(ix) $S = \left\{\frac{4n+3}{n} : n \in N\right\}$,

(x) $S = \left\{\sin\frac{\pi}{6}, \sin\frac{2\pi}{6}, \sin\frac{3\pi}{6}, ..., \sin\frac{n\pi}{6}, ...\right\}$.

Find the supremum and infimum of the above sets, if they exist.

Sol. (i) A is bounded below by 3, but is not bounded above inf $A = 3 \in A$ but sup A does not exist.

(ii) B is bounded, as $1 < x \le 2 \ \forall \ x \in B$.

inf $B = 1 \notin B$, sup $B = 2 \in B$.

(iii) $C = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, ...\right\}$.

C is bounded, as $0 < x \le 1 \ \forall \ x \in C$.

inf $C = 0 \notin C$, sup $C = 1 \in C$.

(iv) D is neither bounded above nor bounded below.

(v) $E = \left\{0, \frac{3}{2}, \frac{2}{3}, \frac{5}{4}, \frac{4}{5}, \frac{7}{6}, \frac{6}{7}, ...\right\}$.

E is bounded, as $0 \le x \le \frac{3}{2} \ \forall \ x \in E$.

inf $E = 0 \in E$, sup $E = \frac{3}{2} \in E$.

(vi) $F = \left\{-1, -\frac{1}{2}, -\frac{1}{3}, -\frac{1}{4}, ...\right\}$

F is bounded, since $-1 \le x \le -0 \ \forall \ x \in F$.

inf $F = -1 \in F$, sup $F = -0 \in F$.

(vii) $G = \left\{-2, -\frac{3}{2}, -\frac{4}{3}, -\frac{5}{4}, ...\right\}$.

G is bounded, since $-2 \le x \le -1 \ \forall \ x \in G$.

inf $G = -2 \in G$. sup $G = -1 \notin G$.

(viii) $S = \left\{-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, -\frac{1}{5}, ...\right\}$ is bounded.

inf S = –1 ∈ S. sup S = $\frac{1}{2}$ ∈ S.

(ix) $S = \left\{7, 5\frac{1}{2}, 5, 4\frac{3}{4}, 4\frac{3}{5}, 4\frac{3}{6}, \ldots\right\}$ is bounded.

sup S = 7 ∈ S, inf S = 4 ∉ S.

(x) $T = \left\{\frac{1}{2}, \frac{\sqrt{3}}{2}, 1, 0, -\frac{1}{2}, -\frac{\sqrt{3}}{2}, -1, \ldots\right\}$ is bounded.

sup T = 1 ∈ T, inf T = –1 ∈ T.

Ex. Find the supremum and infimum of the set :

$$S = \left\{1, 1+\frac{1}{2}, 1+\frac{1}{2}+\frac{1}{2^2}, \ldots, 1+\frac{1}{2}+\frac{1}{2^2}+\ldots+\frac{1}{2^{n-1}}, \ldots\right\}.$$

Sol. We have

$$1+\frac{1}{2}+\frac{1}{2^2}+\ldots+\frac{1}{2^{n-1}}$$

$$= \frac{1\left(1-\frac{1}{2^n}\right)}{1-\frac{1}{2}} = 2\left(1-\frac{1}{2^n}\right)$$

$$= 2-\frac{1}{2^{n-1}} < 2 \ \forall\ n \in N.$$

Hence sup S = 2, inf S = 1.

Ex. The supremum of a non-empty set S, if it exists, is always unique.

Sol. Let, if possible, k, k' be two suprema of S.

Now k' = sup S and k is an upper bound of S ⇒ k' ≤ k. ...(i)

Again k = sup S and k' is an upper bound of S ⇒ k ≤ k'. ...(ii)

From (i) and (ii), k = k'.

Hence the set S has the unique supremum.

Ex. Prove that the greatest member of a set, if it exists, is the supremum (l.u.b.) of the set.

Sol. An element G of a set S is called the greatest member of S if

(i) G∈S and

(ii) x ≤ G ∀ x ∈ S.

Let G be the greatest member of S.

Then x ≤ G ∀ x ∈ S and G is an upper bound of S.

Let G_1 be any number < G, G ∈ S. Then G_1 cannot be an upper bound of S.

(∵ G_1 < G and G ∈ S). Hence G is the l.u.b. (supremum) of S.

Ex. Show that the smallest member of a set, if it exists is the infimum (g.l.b) of the set.

Sol. An element g of a set S is called the smallest member of S if

(i) g ∈ S and

(ii) g ≤ x ∀ x ∈ S.

Let g be the smallest member of a set S.

Then $g \in S$ and $g \le x \ \forall\ x \in S \Rightarrow g$ is a lower bound of S.

Let g_1 be any number > g.

Then $g_1 > g$ and $g \in S \Rightarrow g_1$ cannot be a lower bound of S.

Hence g is the g.l.b. (infimum) of the set S.

Ex. Define the notions supremum and infimum and distinguish them from the greatest and the least points. Give an example of a set of irrationals that has a rational supremum.

Sol. The infimum and supremum of a set may or may not belong to the set but the greatest and least points of a set must belong to the set. Indeed if the greatest and least points of a set S exist, then there are the supremum and infimum of S respectively.

The set $S = \{\sqrt{n+1} - \sqrt{n} : n \in N\}$

i.e., $S = \{\sqrt{2} - \sqrt{1}, \sqrt{3} - \sqrt{2}, \sqrt{4} - \sqrt{3}, ..1\}$

is a set of irrational members having $1 \in Q$ as the supremum.

Notice that $\sqrt{n+1} - \sqrt{n} < 1 \ \forall\ n \in N$, since

$$(1+\sqrt{n})^2 - (\sqrt{n+1})^2 = 2\sqrt{n} > 0 \quad \forall \quad n \in N$$

$$\Rightarrow \quad 1+\sqrt{n} > \sqrt{n+1} \Rightarrow 1 > \sqrt{n+1} - \sqrt{n} \quad \forall \quad n \in N$$

3. COMPLETENESS OF R

Properties of R as a Complete Ordered Field

The set R of real numbers satisfies the following properties :

A_1	$a \in \mathbf{R}$ and $b \in \mathbf{R} \Rightarrow a + b \in \mathbf{R}$.	(Closure law w.r.t. +)
A_2	$a + b = b + a \ \forall\ a, b \in \mathbf{R}$.	(Commutative law w.r.t. +)
A_3	$(a + b) + c = a + (b + c) \ \forall\ a, b, c \in \mathbf{R}$.	(Associative law w.r.t. +)
A_4	$a + 0 = a \ \forall\ a \in \mathbf{R}$.	(Existence of additive identity)
A_5	To each $a \in \mathbf{R}$, $\exists - a \in \mathbf{R}$ such that $a + (- a) = 0$.	(Existence of additive inverse)
M_1	$a \in \mathbf{R}$ and $b \in \mathbf{R} \Rightarrow ab \in \mathbf{R}$	(Closure law w.r.t. multiplication)
M_2	$ab = ba \ \forall\ a, b \in \mathbf{R}$.	(Commutative law w.r.t. multiplication)
M_3	$(ab)\, c = a\, (bc) \ \forall\ a, b, c \in \mathbf{R}$.	(Associative law w.r.t. multiplication)
M_4	$a \,.\, 1 = a \ \forall\ a \in \mathbf{R}$.	(Existence of multiplicative identity)
M_5	For each $0 \neq a \in \mathbf{R}$, $\exists\ a^{-1} \in \mathbf{R}$ such that $aa^{-1} = 1$.	(Existence of multiplicative inverse)
D_L	$a\,(b + c) = ab + ac \ \forall\ a, b, c \in \mathbf{R}$	(Left Distributive law)
D_R	$(b + c)\, a = ba + ca \ \forall\ a, b, c \in \mathbf{R}$	(Right Distributive law)
O_1	For any $a, b \in \mathbf{R}$, Exactly one of the following is true : (i) $a > b$ (ii) $a = b$ (iii) $b > a$.	(Trichotomy law)
O_2	$a > b$ and $b > c \Rightarrow a > c \ \forall\ a, b, c \in \mathbf{R}$.	(Transitive law)
O_3	$a > b \Rightarrow a + c > b + c \ \ \forall\ a, b, c \in \mathbf{R}$.	
O_4	$a > b$ and $c > 0 \Rightarrow ac > bc$.	

The Order Completeness (O.C.) property of **R** :

Every non-empty set of real numbers which is bounded above, has the supremum in **R**.

The set R real numbers is a complete ordered field

The set of rational numbers is not order complete.

Example

Let $S = \{x : x \in \mathbf{Q}.\ x > 0 \text{ and } x^2 < 2\}$.

$S \neq \phi$ ($\because 1 \in S$) and 2 is an upper bound of S.

S is a non-empty bounded above subset of **Q**.

The Order Completeness Axioms [OC]

[OC_1] The Lub Axiom : Every non-empty set S of ordered field F which is bounded above admits of (lub) supremum in F.

[OC_2] The Glb Axiom : Every non-empty set S of ordered field F which is bounded below admits of (lub) supremum in F.

Thus, glb axioms $\Leftrightarrow$ lub axioms

Complete Ordered Field

Definition : The ordered field F which when satisfies the above completeness axioms is called **complete ordered field.**

Explicit Statement of the properties of Real numbers which characterize the real number system as a complete ordered field :

1. **The Field Axioms :** A_1 to A_5 and M_1 to M_5 and D_L, D_R.

2. **Order Axioms (>) : O_1 to O_4**
3. **Completeness Axioms :** The lub axiom

We shall call this complete ordered field as a field of real numbers and shall continue to denote by **R**.

Another Equivalent Completeness Axioms

Dedekind's Completeness Axioms : If all the real numbers be divided into two subsets (classes) L and U of **R** then,

DC_1 : Each subset has atleast one real number.

i.e., $L \neq \phi$, $U \neq \phi$

DC_2 : Each real number either belongs to L or U.

i.e., $L \cup U = \mathbf{R}$

DC_3 : Each member (real number) of class **L** is smaller than every member (real number) of class U,

i.e., if $x \in L$, $y \in U \Rightarrow x < y$.

then either the subset L has a greatest member or the subset U has a least member.

From the above axioms, complete ordered axioms can be deduced and conversely. These are equivalent.

Ordered complete axiom $\Leftrightarrow$ Dedekind axiom

The Complete ordered field is also called Linear continuum and this is unique.

Let R_1 and R_2 be two complete ordered fields then we can always define one-one bijection between R_1 and R_2 such that $\forall$ x, y $\in$ R.

$$f(x + y) = f(x) + f(y), \qquad f(x), f(y) \in R_2$$

and $$f(xy) = f(x) \,.\, f(y)$$

Clearly, $$f : R_1 \to R_2$$

Here zero and identity elements of R_1 are mapped on zero and identity elements of R_2 respectively. This mapping is called Isomorphic mapping and R_1 and R_2 are called one-one isomorphic i.e., $R_1 \cong R_2$ whose structure is also same. There may be difference of notation only.

Therefore if $f(x) = x$, then both the complete ordered field become same i.e., the complete ordered field is unique.

Theorem : The set **Q** of rational numbers is not a complete ordered field.

Theorem : A non-empty subset of real numbers which is bounded below has the greatest lower bound (infimum) in **R**.

Proof. Let S be any non-empty bounded below subset of **R** and k be its lower bound.

then, $k \leq x \ \forall \ x \in S$.

Let $T = \{-x : x \in S\}$. Then $T \neq \phi$, as $S \neq \phi$.

We shall prove that T is bounded above. We see that

$$k \leq x \Rightarrow -k \geq -x \ \forall \ x \in S$$

$\Rightarrow$ $(-x) \leq -k \ \forall \ x \in S$.

Thus T is bounded above with $-k$ as its upper bound.

Since T is non-empty and bounded above, by order completeness property of **R,** T has supremum in **R.**

Let K = sup T.

$\Rightarrow$ $(-x) \leq K \ \forall \ x \in S \Rightarrow x \geq -K \ \forall \ x \in S$.

$\Rightarrow$ $(-K)$ is a lower bound of S.

We shall now show that $(-K) = \inf S$.

Let k' be any lower bound of S. As shown above, $-k'$ is an upper bound of T. Since $K = \sup T$ $\Rightarrow k \leq -k' \Rightarrow -K \geq k' \Rightarrow -K = \inf S$.

Hence S has the infimum in **R**.

Theorem : If S a non-empty set of real numbers which is bounded above, then a real numbers is the supremum of S if and only if the following two conditions holds

(i) $x \leq s \ \forall \ x \in S$.

(ii) Given any $\varepsilon > 0$, $\exists$ some $x \in S$ such that $x > s - \varepsilon$.

Proof. The condition is necessary.

Let $s = \sup S = \text{l.u.b. } S$.

By definition, $x \leq s$, $\forall \ x \in S$.

This proves condition (i).

Let $\varepsilon > 0$.

Then $s - \varepsilon < s$ cannot be an upper bound of S.

So there exists some $x \in S$ such that $x > s - \varepsilon$.

This proves condition (ii).

The condition is sufficient.

Let s be any real number satisfying conditions (i) and (ii).

We shall prove that $s = \sup S$.

Now $x \leq s \ \forall \ x \ S \Rightarrow s$ is an upper bound of S.

Let, if possible s' be any real number $< s \Rightarrow s - s' > 0$.

Let $\varepsilon = s - s' > 0$.

By condition (ii) $\exists$ some $x \in S$ such that $x > s - \varepsilon = s - (s - s') = s'$

$\Rightarrow$ $x > s'$ for some $x \in S$.

$\Rightarrow$ s' is not an upper bound of S.

We see that no real number $< s$ can be an upper bound of S.

Hence $s = \sup S$.

Theorem : If S is a non-empty subset of **R** which is bounded below, then a real number t is the infimum of S iff the following two conditions hold :

(i) $x \geq t \ \forall \ x \in S$.

(ii) Given any $\varepsilon > 0$, $\exists$ some $x \in S$ such that $x < t + \varepsilon$.

Proof. The condition is necessary.

Let $t = \inf S = \text{g.l.b. } S$.

$\Rightarrow$ $x \geq t \ \forall \ x \in S$, which proves condition (i).

Let $\varepsilon > 0$ then, $t + \varepsilon > t$ cannot be a lower bound of S.

So there exists some $x \in S$ such that $x < t + \varepsilon$. This proves condition (ii).

The condition is sufficient.

Let t be any real number satisfying conditions (i) and (ii).

We shall prove that $t = \inf S$.

Now $x \geq t \ \forall \ x \in S \Rightarrow t$ is a lower bound of S.

Let if possible t' be any real number $> t \Rightarrow t' - t > 0$.

Let $\varepsilon = t' - t > 0$.

By condition (ii) $\exists$ some $x \in S$ such that $x < t + \varepsilon$

$\Rightarrow \quad x < t + (t' - t) = t' \Rightarrow x < t'$ for some $x \in S \Rightarrow t'$ is not a lower bound of S.

We see that no real number > t can be a lower bound of S.

Hence t = inf S.

Example. Let S be a non-empty subset of **R** and $s \in$ **R.** Then s = sup S iff

(i) $x \le s \ \forall \ x \in S$

(ii) For each $s' \in$ **R** such that $s' < s$, $\exists$ some $x \in S$ such that $s' < x$.

Example. Let S be a non-empty subset of **R** and $t \in$ **R.** Then t = inf S iff

(i) $x \ge t \ \forall \ x \in S$

(ii) For each $t' \in$ **R** such that $t' > t$, $\exists$ some $x \in S$ such that $t' > x$.

Representation of Real Numbers as Points on a Straight Line

The points on a straight line can be used to represent real numbers. Let X' X be any straight line. Mark two points O and A on it such that A is to the right of O. The point O divides the straight line into two parts. We call the part that lies to the right of O as the positive part, and the part that lies to the left of O the negative part. Such a line for which positive and negative sides are fixed is called a directed line.

Suppose the points O and A represent the numbers 0 and 1, respectively. With OA as unity on a certain scale, we can locate all positive integers 1, 2, 3... on the right of O and all negative integers ..., –3, –2, –1 on the left of O. To represent any positive rational number m/n, we locate a point P on the right of O such that OP is m times the nth part of the unit length OA. Similarly, all negative rational numbers can be located on the left of O. The point on the straight line corresponding to rational numbers are called rational points. It might appear that rational points cover the entire straight line. But this is not true. There exist infinite number of points on the line which are not rational.

For Example, the point Q on the line such that OQ is equal to the diagonal of the square with side OA (OQ = $\sqrt{2}$) does not correspond to any rational number. Such points on the line which are not rational indeed fill up the gaps between the rational points are called irrational points and these correspond to irrational numbers. In this way every real number can be represented as a point on the straight line. On account of this representation, we refer to a real number as a point of **R** and the straight line is referred to as **real line.**

Dedekind-Cantor Axiom

To every real number corresponds a unique point on a directed line and to every point on a directed line corresponds a unique real number.

Absolute Value of Modulus of a Real Number

Definition. Let a be any real number. The **absolute value** or **modulus** of a, denoted by | a |, is defined as

$$|a| = \begin{cases} a \text{ if } a \ge 0 \\ -a \text{ if } a < 0 \end{cases}$$

Remarks

1. $|a| \ge 0 \ \forall \ a \in \mathbb{R}$.
2. $|a| = |-a|$.

Theorem : If a is any real number, then $|a| = \max\{a, -a\}$.

Proof. We have

$$|a| = a, \text{ if } a \ge 0, \qquad \ldots(1)$$

and $$|a| = -a > a, \text{ if } a < 0. \qquad \ldots(2)$$

From (1) and (2), we get $|a| \ge a$.

Again $\quad |a| = a \geq -a$, if $a \geq 0$, ...(3)

and $\quad |a| = -a$, if $a < 0$. ...(4)

From (3) and (4), we get $|a| \geq -a$.

Hence $|a|$ is the maximum of the two real numbers a and $-a$ i.e.,

$$|a| = \max \{a, -a.\}$$

Theorem . If $x, y \in \mathbf{R}$; then

(i) $|x|^2 = x^2 = |-x|^2$.

(ii) $|xy| = |x| . |y|$.

(iii) $\left|\dfrac{x}{y}\right| = \dfrac{|x|}{|y|}, y \neq 0$.

(iv) $|x + y|^2 + |x - y|^2 = 2 (|x|^2 + |y|^2)$.

Proof. (i) $\quad$ if $x \geq 0$ then $|x| = x \Rightarrow |x|^2 = x^2$,

and $\quad$ if $x < 0$, then $|x| = -x \Rightarrow |x|^2 = (-x)^2 = x^2$.

Thus $\quad |x|^2 = x^2$.

Similarly, we can prove that $|-x|^2 = x^2$.

(ii) Using (i), $|xy|^2 = (xy)^2 = x^2y^2 = |x|^2.|y|^2 = (|x|.|y|)^2$

$\therefore \quad |xy| = \pm (|x|.|y|)$.

Since $|x|$ is non-negative $\forall\, x \in \mathbf{R}$, therefore

$$|xy| = |x|.|y|.$$

(iii) Using (i) $\left|\dfrac{x}{y}\right|^2 = \left(\dfrac{x}{y}\right)^2 = \dfrac{x^2}{y^2} = \dfrac{|x|^2}{|y|^2} = \left(\dfrac{|x|}{|y|}\right)^2$

$\therefore \quad \left|\dfrac{x}{y}\right| = \pm\dfrac{|x|}{|y|}$.

Since $|x|$ is non-negative $\forall\, x \in \mathbf{R}$, therefore

$$\left|\frac{x}{y}\right| = \frac{|x|}{|y|}, y \neq 0.$$

(iv) $|x + y|^2 + |x - y|^2 = (x + y)^2 + (x - y)^2$, by part (i)

$= 2 (x^2 + y^2)$

$= 2 (|x|^2 + |y|^2)$, by part (i)

Theorem . If x, y are any two real numbers, then

(i) $|x + y| \leq |x| + |y|$ (Triangle Inequality)

(ii) $|x - y| \geq ||x| - |y||$.

(iii) $|x - y| \leq |x| + |y|$.

Proof. (i) To show $|x + y| \leq |x| + |y|$.

Case I. Let $x + y \geq 0 \Rightarrow |x + y| = x + y$, by definition

$\leq |x| + |y| \quad (\because x \leq |x|\ \forall\, x \in \mathbf{R})$

$\therefore \quad |x + y| \leq |x| + |y|$.

Case II. Let $x + y < 0 \Rightarrow |x + y| = -(x + y) = (-x) + (-y)$.

Since $-x \le |x| \ \forall \ x \in \mathbf{R}$, $\therefore$ $|x + y| \le |x| + |y|$.

Hence $|x + y| \le |x| + |y|$.

Aliter. We know

$$|x + y|^2 = (x + y)^2 = x^2 + y^2 + 2xy = |x|^2 + |y|^2 + 2xy$$

or $\quad |x + y|^2 \le |x|^2 + |y|^2 + 2\,|x|\,|y| \qquad (\because xy \le |xy| = |x|\,|y|)$

$\therefore \quad |x + y|^2 \le (|x| + |y|)^2$.

Hence $|x + y| \le |x| + |y|$, since modulus is non-negative.

(ii) To show $|x - y| \ge ||x| - |y||$.

We have $\quad |x| = |x - y + y \le |x - y| + |y|$

$\Rightarrow \quad |x| - |y| \le |x - y|$. ...(1)

Again $|y| = |y - x + x| \le |y - x| + |x|$

$\Rightarrow \quad |y| - |x| \le |-(x - y)|$

$\Rightarrow \quad -(|x| - |y|) \le |x - y|$. ...(2)

From (1) and (2), $|x - y| \ge |x| - |y|$,

and $\quad |x - y| \ge -(|x| - |y|)$.

$\therefore \quad |x - y| \ge \max \{(|x| - |y|), -(|x| - |y|)\}$.

Hence $\quad |x - y| \ge ||x| - |y||$.

$(\because \max \{a, -a\} = |a| \ \forall \ a \in \mathbf{R})$

(iii) $|x - y| = |x + (-y)| \le |x| + |-y| = |x| + |y|$.

Hence $\quad |x - y| \le |x| + |y|$.

Ex. If x and y are real numbers and $\varepsilon > 0$,

(i) Show that $|x - y| < \varepsilon$ if and only if $y - \varepsilon < x < y + \varepsilon$.

(ii) Show that $|x| < a \Leftrightarrow -a < x < a$.

Sol. (i) We know $|a| = \max \{a, -a\} \ \forall \ a \in \mathbf{R}$.

$\therefore \quad |x - y| = \max \{(x - y), -(x - y)\} = \max \{x - y, y - x\}$.

It follows that $|x - y| < \varepsilon$

$\Leftrightarrow \quad \max \{x - y, y - x\} < \varepsilon$

$\Leftrightarrow \quad x - y < \varepsilon$ and $y - x < \varepsilon$

$\Leftrightarrow \quad x < y + \varepsilon$ and $y - \varepsilon < x$

$\Leftrightarrow \quad y - \varepsilon < x < y + \varepsilon$.

(ii) $|x| = \max \{x, -x\} < a$

$\Leftrightarrow \quad x < a$ and $-x < a$

$\Leftrightarrow \quad x < a$ and $x > -a$

$\Leftrightarrow \quad -a < x$ and $x < a$

$\Leftrightarrow \quad -a < x < a$.

Ex. Show that for all real numbers x and y,

$$\frac{|x+y|}{1+|x+y|} \le \frac{|x|}{1+|x|} + \frac{|y|}{1+|y|}.$$

Sol. We know $|x + y| \le |x| + |y|$

$\Rightarrow \quad 1 + |x + y| \le 1 + |x| + |y|$

$$\Rightarrow \quad \frac{1}{1+|x+y|} \ge \frac{1}{1+|x|+|y|} \Rightarrow -\frac{1}{1+|x+y|} \le -\frac{1}{1+|x|+|y|}$$

$$\Rightarrow \quad 1-\frac{1}{1+|x+y|} \le 1-\frac{1}{1+|x|+|y|} \Rightarrow \frac{|x+y|}{1+|x+y|} \le \frac{|x|+|y|}{1+|x|+|y|}$$

$$\Rightarrow \quad \frac{|x+y|}{1+|x+y|} \le \frac{|x|}{1+|x|+|y|} + \frac{|y|}{1+|x|+|y|} \le \frac{|x|}{1+|x|} + \frac{|y|}{1+|y|}.$$

$(\because 1 + |x| + |y| \ge 1 + |x|$ and $1 + |x| + |y| \ge 1 + |y|)$

Ex. If $x_1, x_2, \ldots, x_n$ are real numbers, then show that

(i) $|x_1x_2 \ldots x_n| = |x_1|\,|x_2| \ldots |x_n|$.

(ii) $|x_1 + x_2 + \ldots + x_n| \le |x_1| + |x_2| + \ldots + |x_n|$.

Sol. (i) $|x_1x_2 \ldots x_n| = |x_1|\,|x_2x_3 \ldots x_n)$ $\quad (\because |xy| = |x|\,|y|)$

$= |x_1|\,|x_2|\,|x_3 \ldots x_n|$, and so on

Thus $|x_1x_2 \ldots x_n| = |x_1|\,|x_2| \ldots |x_n|$.

(ii) $|x_1 + x_2 + \ldots + x_n| = |x_1 + (x_2\ x_3 + \ldots + x_n)|$

$\le |x_1| + |x_2 + x_3 + \ldots + x_n|$

$(\because |x + y| \le |x| + |y|)$

$\le |x_1| + |x_2| + x_3 + \ldots + x_n|$, and so on

Thus $|x_1 + x_2 + \ldots + x_n| \le |x_1| + |x_2| + \ldots + |x_n|$.

Ex. Show that

(i) $|x + y| \ge 1 \Rightarrow |x| \ge \frac{1}{2}$ or $|y| \ge \frac{1}{2}$.

(ii) $|x_1 + x_2 + \ldots + x_n| \ge 1 \Rightarrow |x_i| \ge \frac{1}{n}$ for some i with $1 \le i \le n$.

Sol. (i) Suppose the conclusion is false. Then

$$|x| < \frac{1}{2} \text{ and } |y| < \frac{1}{2}.$$

Now $|x + y| \le |x| + |y| < \frac{1}{2} + \frac{1}{2} = 1 \Rightarrow |x + y| < 1$, a contradiction.

Hence $|x| \ge \frac{1}{2}$ or $|y| \ge \frac{1}{2}$.

(ii) Suppose the conclusion is false. Then

$$|x_i| < \frac{1}{n}, \text{ for each i with } 1 \le i \le n.$$

Now $|x_1 + x_2 + \ldots + x_n| \le |x_1| + |x_2| + \ldots + |x_n|$

$$< \frac{1}{n} + \frac{1}{n} + \ldots + \frac{1}{n} = n.\frac{1}{n} = 1$$

$\therefore$ $|x_1 + x_2 + \ldots + x_n| < 1$, a contradiction.

Hence $|x_i| \ge \frac{1}{n}$, for some i with $1 \le i \le n$.

4. LIMIT POINTS

Neighbourhood of a Point

Definition : A set $S \subset \mathbf{R}$ is called a **neighbourhood (nbd)** of a point $p \in \mathbf{R}$ if there exists an open interval $(p - \varepsilon, p + \varepsilon)$ for some $\varepsilon > 0$,

such that

$$p \in (p - \varepsilon, p + \varepsilon) \subset S.$$

Examples

1. The set **R** of real numbers is nbd. of all its points.

   For any real number x, we have

   $$x \in (x - \varepsilon, x + \varepsilon) \subset \mathbf{R}, \quad \varepsilon > 0.$$

2. The open interval (a, b) is a nbd. of all its points, since

   $$x \in (a + \varepsilon, b - \varepsilon) \subset (a,b)$$

3. The half open interval (a, b] is a nbd, of all its points except b, since for any $\varepsilon > 0$

   $$b \in [b - \varepsilon, b + \varepsilon) \not\subset (a, b]. \qquad (\because b + \varepsilon > b)$$

   Similarly, [a, b) is a nbd. of all its points except a and [a, b] is a nbd. of all its points except a and b.

4. The set **Z** of integers is not a nbd. of any of its points, since for any integer p and any $\varepsilon > 0$,

   $$p \in (p - \varepsilon, p + \varepsilon) \not\subset \mathbf{Z}.$$

5. Each of the following open intervals.

   $$(-1, 1), \left(-\frac{1}{2}, \frac{1}{2}\right), \left(-\frac{1}{3}, \frac{1}{3}\right), \ldots, \left(-\frac{1}{n}, \frac{1}{n}\right), \ldots \text{ is a nbd of } 0.$$

6. For any $p \in \mathbf{R}$, $\mathbf{N} = \{p\}$ is not a a nbd. of p, since for any $\varepsilon > 0$, $(p - \varepsilon, p + \varepsilon) \not\subset \mathbf{N}$.

7. The set **Q** of rational numbers is not a nbd. of any of its points, for if (a, b) is any open interval around a point $p \in \mathbf{Q}$, then

   $$p \in (a, b) \not\subset \mathbf{Q}$$

   since between any two real numbers a, b there is always an irrational number which does not belong to **Q**.

8. The set $\mathbf{R} \sim \mathbf{Q}$ of irrational numbers is not a nbd. of any of its points.

9. A non-empty finite set S is not a nbd. of any of its points, if (a, b) is any open interval around any point $x \in S$, then

   $$x \in (a, b) \not\subset S,$$

   since (a, b) necessarily contains infinite number of points and S is a finite set.

10. The empty set ϕ is a nbd. of each of its points, since there is no point in ϕ which is not a nbd.

Ex. If M is a nbd. of a point p and $N \supset M$, then N is also a nbd. of the point p.

Sol. Since M is a nbd. of the point p therefore, there exists an open interval $(p - \varepsilon, p + \varepsilon)$, $\varepsilon > 0$ such that

$$p \in (p - \varepsilon, p + \varepsilon) \in M$$

$$\Rightarrow \quad p \in (p - \varepsilon, p + \varepsilon) \subset M \subset N \qquad [\because N \supset M \Rightarrow M \subset N]$$

$$\Rightarrow \quad p \in (p - \varepsilon, p + \varepsilon) \subset N.$$

Hence N is a nbd. of p.

Ex. If M and N are nbds. of a point p, then $M \cap N$ is also a nbd. of p.

Sol. Since M and N are nbds. of p, therefore, there exist $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ such that

$$p \in (p - \varepsilon_1, p + \varepsilon_2) \subset M,$$

and $$p \in (p - \varepsilon_2, p + \varepsilon_2) \subset N.$$

Let $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$ so that $\varepsilon \le \varepsilon_1$ and $\varepsilon \le \varepsilon_2$. Thus

$$\left.\begin{array}{l} p \in (p-\varepsilon, p+\varepsilon) \subset (p-\varepsilon_1, p+\varepsilon_1) \subset M, \\ p \in (p-\varepsilon, p+\varepsilon) \subset (p-\varepsilon_2, p+\varepsilon_2) \subset N. \end{array}\right\} \quad ...(1)$$

$\therefore$ $$p \in (p - \varepsilon, p + \varepsilon) \subset M \cap N.$$

Hence $M \cap N$ is a nbd of p.

Note. Obviously, $p \in (p - \varepsilon, p + \varepsilon) \subset M \cup N$ and so $M \cup N$ is also a nbd. of p.

Ex. Show that intersection of a finite number of neighbourhoods of a point is also a neighbourhood of the point.

Sol. Let $M_1, M_2, ..., M_n$ be a finite number of neighbourhoods of a point p. Then there exists $\varepsilon_i > 0$ such that

$$p \in \,]\, p - \varepsilon_i, p + \varepsilon_i \,[\, \subset M_i \text{ for } i = 1, 2, ..., n.$$

Let $\varepsilon = \min\{\varepsilon_1, \varepsilon_2, ..., \varepsilon_n\}$. Then

$$p \in \,]\, p - \varepsilon, p + \varepsilon \,[\, \subset M_i \text{ for } i = 1, 2, ..., n$$

$$\Rightarrow \quad p \in \,]\, p - \varepsilon, p + \varepsilon \,[\, \subset M_1 \cap M_2 \cap ... \cap M_n.$$

Hence $M_1 \cap M_2 \cap ... \cap M_n$ is a nbd. of p.

Ex. Define a neighbourhood of a point. Show that the intersection of the family of all neighbourhoods of a point x is {x}.

Sol. A subset $S \subset R$ is called a neighbourhood of a point $p \in R$ if there exists some $\varepsilon > 0$ such that $(p - \varepsilon, p + \varepsilon) \subset S$.

Let $\{N_x\}$ be a family of all neighbourhoods of a point x. We have to show that $\cap N_x = \{x\}$.

Consider any point $y \ne x$ and Let $\varepsilon = |x - y|$

then, $(x - \varepsilon, x + \varepsilon)$ is a nbd. of x and $y \notin (x - \varepsilon, x + \varepsilon)$,

since $y \in (x - \varepsilon, x + \varepsilon) \Rightarrow x - \varepsilon < y < x + \varepsilon \Rightarrow |x - y| < \varepsilon$, a contradiction.

Since $y \notin (x - \varepsilon, x + \varepsilon)$, $y \notin \cap N_x$. Hence $\cap N_x = \{x\}$.

Ex. (i) Can a set **N** be a neighbourhood of any point of **R** ~ **N** ?

(ii) Can a non-empty finite set be a neighbourhood of any of its points ?

Sol. (i) We shall show that **N** cannot be a neighbourhood of any point of **R** ~ **N**.

Let $x \in$ **R** ~ **N** then, $x \notin$ **N**.

Suppose **N** is a nbd. of x then there exists some $\varepsilon > 0$ such that

$x \in (x - \varepsilon, x + \varepsilon) \subset$ **N** $\Rightarrow x \notin$ N, which is a contradiction.

Hence the assertion made above is not TRUE.

(ii) The answer of the second part is also No.

Let $x \in$ **N**, where N is a non-empty finite set. If **N** is nbd. of x, then

$x \in (x - \varepsilon, x + \varepsilon) \subset$ **N** for some $\varepsilon > 0$.

This is impossible, since N is a finite set and $(x - \varepsilon, x + \varepsilon)$ contains infinite number of elements.

Limit Points

Definition 1 : A real number p is said to be a **limit point** of a set $S \subset$ **R** if for each $\varepsilon > 0$ there exists at least one point $p_1 \in S$ such that

$$p_1 \in (p - \varepsilon, p + \varepsilon) \text{ and } p_1 \ne p.$$

Definition 2 : The set of all limit points of a set $S \subset \mathbf{R}$ is called the **derived set** of S and is denoted as S'.

$x \in S' \Leftrightarrow x$ is a limit point of S.

Example

1. 0 is the only limit point of the set

$$S = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{n}, \ldots\right\}$$

2. 1 and –1 are the only limit points of the set

$$S = \left\{1, -1, 1\frac{1}{2}, -1\frac{1}{2}, 1\frac{1}{3}, -1\frac{1}{3}, \ldots\right\}.$$

Notice that 1, –1 both belong to S and S' = {1, –1}.

Remarks

- A set may or may not have a limit point.
  For Example, **R** has limit points but **Z** has no limit point.
- Limit point of a set may or may not belong to the set.
  For Example, every limit point of **R** belongs to **R** and the only limit point 0 of the set $S = \{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$ does not belong to the set i.e., $0 \notin S'$.

Examples :

1. The set I has no limit point for a nbd $\left(m - \frac{1}{2}, m + \frac{1}{2}\right)$ of $m \in I$ contains no point of I other than m. Thus the derived set of I is the null set Ø.
2. Every point of **R** is a limit point for every nbd of any of its points contains an infinity of members of **R**. Therefore, **R' = R**.

Ex. Find the limit points of each of the following sets :

(a) $A = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right\}$ (b) $B = \left\{1 + \frac{1}{n} : n \in N\right\}$

(c) $C = \left\{1, -1, 1\frac{1}{2}, -1\frac{1}{2}, 1\frac{1}{3}, -1\frac{1}{3}, \ldots\right\}$ (d) $D = \{x : 0 \le x < 1\}$

(d) $D = \{x : 0 \le x < 1\}$. (e) $E = \left\{\frac{1}{m} + \frac{1}{n} : m \in N, n \in N\right\}$

(f) $F = \{3^{-n} + 5^{-n} : n \in N\}$

Sol. (a) The only limit point of A is 0 i.e., A' = {0}. Notice that $0 \notin A$.

(b) We have $B = \left\{2, 1\frac{1}{2}, 1\frac{1}{3}, 1\frac{1}{4}, \ldots\right\}$.

The only limit point of B is 1 i.e., B' = {1}. Notice that $1 \notin B$.

(c) The set $C = \left\{1, -1, 1\frac{1}{2}, -1\frac{1}{2}, 1\frac{1}{3}, -1\frac{1}{3}, \ldots\right\}$

has only two limit points 1, –1. Notice that $1 \in C$ and $-1 \in C$.

We have C' = {1, –1}.

(d) We observe that D = [0, 1) and so D' = [0, 1].

(e) First we write all the elements of the set

$$E = \left\{\frac{1}{m} + \frac{1}{n} : m, n \in N\right\}$$

For m = 1, the elements of E are :

$$1+1, 1+\frac{1}{2}, 1+\frac{1}{3}, ... \quad ...(1)$$

For m = 2, the elements of E are :

$$\frac{1}{2}+1, \frac{1}{2}+\frac{1}{2}, \frac{1}{2}+\frac{1}{3}, \frac{1}{2}+\frac{1}{4}, ... \quad ...(2)$$

For m = 3, the elements of E are

$$\frac{1}{3}+1, \frac{1}{3}+\frac{1}{2}, \frac{1}{3}+\frac{1}{3}, \frac{1}{3}+\frac{1}{4}, ... \quad ...(3)$$

and so on.

It is clear that

1 is the limit point of the elements in row (1),

$\frac{1}{2}$ is the limit point of the elements is row (2),

$\frac{1}{3}$ is the limit point of the elements in row (3),

and so on.

Thus $1, \frac{1}{2}, \frac{1}{3}, ..., \frac{1}{n}, ...$ are all limit points of E.

Since E contains an infinite number of fractions lying between 0 and 1, so each nbd. of 0 surely contains a positive fraction and as such 0 is also a limit point of E.

Hence $E' = \left\{\frac{1}{n} : n \in N\right\} \cup \{0\}$.

(f) We have $F = \left\{\frac{1}{3^n} + \frac{1}{5^n} : n \in N\right\}$.

As discussed in part (e), we observe that

$$F' = \left\{\frac{1}{3}, \frac{1}{3^2}, \frac{1}{3^3}, ...\right\} \cup \left\{\frac{1}{5}, \frac{1}{5^2}, \frac{1}{5^3}, ...\right\} \cup \{0\}.$$

Example. Give an example of a set whose derived set is

(i) void

Ans. If **Z** is the set of integers, then **Z'** = ϕ.

(ii) subset of the given set

Ans. Let S = $\left\{1,-1,1\frac{1}{2},-1\frac{1}{2},1\frac{1}{3},-\frac{1}{3},-\frac{1}{3},...\right\}$.

Then S' = {1, –1} $\subset$ S.

(iii) superset of the given set

Ans. Let S = $\left\{1,\frac{1}{2},\frac{1}{3},...\frac{1}{n},...\right\}$. Then S' = {0}.

S' is neither a subset nor a superset of S.

(v) same as the given set

Ans. **R**' = **R** or [a, b]' = [a, b].

Example : Give an example of a set which has

(i) no limit point

Ans. The set **Z** of integers has no limit point.

(ii) unique limit point

Ans. S = $\left\{1,\frac{1}{2},\frac{1}{3},...,\frac{1}{n},...\right\}$ has 0 as the unique limit point.

(iii) two limit points

Ans. S = $\left\{1,-1,1\frac{1}{2},-1\frac{1}{2},1\frac{1}{3},-1\frac{1}{3},...\right\}$

has two limit points viz 1 and –1.

(iv) three limit points

Ans. S = $\left\{1,2,3,1\frac{1}{2},2\frac{1}{2},\frac{1}{3},2\frac{1}{3},3\frac{1}{3},...\right\}$ has three limit points viz. 1, 2, 3

(v) infinite number of limit points

Ans. The set **R** of real numbers has infinite number of limit points. In fact every real number is a limit point of **R**.

(vi) a set whose every point is a limit points of the set

Ans. The set **R** of real numbers.

(vii) a set none of whose points is a limit point of the set

Ans. The set **Z** of integers.

(viii) A set with only $\sqrt{3}$ as its limit point.

Ans. S = $\left\{\sqrt{3}+\frac{1}{n}:n\in N\right\}$.

Ex. Define limit point of a set. Show that the set **N** of natural numbers has no limit point.

Sol. Let m be any natural number.

Then $\left(m-\frac{1}{2},m+\frac{1}{2}\right)$ is a nbd. of m which does not contain any point of **N** different from m.

Let $m \in \mathbf{R} \sim \mathbf{N}$ i.e., m is any real number but not a natural number. Then we can find some integer j such that $j < m < j + 1$. Consequently $(j, j + 1)$ is a nbd. of m, which does not contain any point of $\mathbf{N}$. Hence $\mathbf{N}$ has no limit point i.e., $\mathbf{N}' = \phi$.

Ex. Given an example each of :

(i) a bounded set of real numbers

(ii) an unbounded set of real numbers

Also find the set of limit points of both the sets.

Sol. **(i)** $S = \left\{1, -1, 1\frac{1}{2}, -1\frac{1}{2}, 1\frac{1}{3}, -1\frac{1}{3}, \ldots\right\}$

is a bounded set of real numbers. Its limit points are 1 and –1.

(ii) $S = \left\{1, 2, \frac{1}{2}, 3, \frac{1}{3}, 4, \frac{1}{4}, \ldots\right\}$

is an unbounded set of real numbers. 0 is the only limit point of S.

Ex. Find the derived sets of the following :

(i) $A = \left\{1 + \frac{(-1)^n}{n} : n \in N\right\}$, (ii) $B = \left\{(-1)^n + \frac{1}{n} : n \in N\right\}$,

(iii) $C = \left\{\frac{1 + (-1)^n}{n} : n \in N\right\}$, (iv) $D = \left\{3^n + \frac{1}{3^n} : n \in N\right\}$

Sol. $A' = \{1\}$, $B' = \{-1, 1\}$, $C' = \{0\}$, $D' = \phi$.

Ex. If S and T are sets of real numbers, then show that

(a) $S \subset T \Rightarrow S' \subset T'$ (b) $(S \cup T)' = S' \cup T'$

(c) $(S \cap T)' \subset S' \cap T'$

Given an example to show that $(S \cap T)'$ and $S' \cap T'$ may not be equal.

Sol. **(a)** Let $S \subset T$. We shall prove that $S' \subset T'$.

Let $x \in S'$ so that x is a limit point of S. Thus for each $\varepsilon > 0$, $(x - \varepsilon, x + \varepsilon)$ contains a point $x_1 \in S$ such that $x_1 \neq x$. Since $S \subset T$, $x_1 \in S \Rightarrow x_1 \in T$.

Consequently, $(x - \varepsilon, x + \varepsilon)$ contains a point $x_1 \in T$, $x_1 \neq x$, and so x is a limit point of T i.e., $x \in T'$.

$\therefore$ $x \in S' \Rightarrow x \in T'$.

Hence $S' \subset T'$.

(b) We know $S \subset S \cup T \Rightarrow S' \subset (S \cup T)'$, by part (a).

Similarly, $T \subset S \cup T \Rightarrow T' \subset (S \cup T)'$.

$\therefore$ $S' \cup T' \subset (S \cup T)'$. ...(1)

Now we prove $(S \cup T)' \subset S' \cup T'$. ...(2)

Let $x \in (S \cup T)' \Rightarrow x$ is a limit point of $S \cup T$

$\Rightarrow$ every nbd. of x contains a point $y \in S \cup T$, $y \neq x$

$\Rightarrow$ every nbd. of x contains a point ($y \in S$ or $y \in T$), $y \neq x$

$\Rightarrow$ every nbd. of x contains a point $y \in S$, $y \neq x$,

or every nbd. of x contains a point $y \in T$, $y \neq x$.

$\Rightarrow$ x is a limit point of S or x is a limit point of T

$\Rightarrow x \in S'$ or $x \in T'$

$\Rightarrow x \in S' \cup T'$.

Thus (2) is proved.

From (1) and (2), $(S \cup T)' = S' \cup T'$.

(c) We know $S \cap T \subset S \quad \Rightarrow \quad (S \cap T)' \subset S'$, by part (a)

and $\quad S \cap T \subset T \quad \Rightarrow \quad (S \cap T)' \subset T'$.

$\therefore \quad (S \cap T)' \subset S' \cap T'$.

To prove that the equality does not hold, we take

$S =]1, 2[$ and $T =]2, 3[$ so that $S \cap T = \phi$.

$\Rightarrow \quad S' = [1, 2]$, $T = [2, 3]$ and

$(S \cap T)' = \phi' = \phi$.

Now $\quad S' \cap T' = \{2\}$ and $(S \cap T)' = \phi$.

Thus $\quad (S \cap T)' \neq S' \cap T'$.

Some Theorems of Limit Points

Theorem : A point p is a limit point of a set A if and only if every neighbourhood of p contains infinitely many points of A.

Proof. The condition is necessary.

Let p be a limit point of A. We shall show that every nbd. of p contains infinitely many points of A. Suppose this is false then, there exists some $\varepsilon > 0$ such that $(p - \varepsilon, p + \varepsilon)$ contains only a finite number of points, say $p_1, p_2, \ldots, p_n$; which are different from p.

Let $\quad \delta = \min \{ | p - p_1 |, | p - p_2 |, \ldots, | p - p_n | \} \; (\delta > 0)$

$\Rightarrow \quad \delta \leq | p - p_i |$ for $i = 1, 2, \ldots, n$

$\Rightarrow \quad (p - \delta, p + \delta)$ contains no point of A other than p

$\Rightarrow \quad$ p is not a limit point of A, which is a contradiction.

Hence every nbd. of p must contain infinitely many points of A.

The condition is sufficient.

Suppose every nbd. of p contains infinitely many points of A. Surely then, every nbd. of p contains a point of A different from p and so p is a limit point of A.

Corollary. Show that a finite set has no limit points.

Proof. Let S be a finite set. Let if possible p be a limit point of S.

By Theorem for every $\varepsilon > 0$, $(p - \varepsilon, p + \varepsilon)$ must contain an infinite number of elements of S, which is impossible, as S is a finite set. Hence p cannot be a limit point of S.

Theorem : (Bolzano-Weierstrass Theorem)

Every infinite bounded set of real numbers has a limit point.

Proof. Let S be any infinite bounded set of real numbers with bounds

$$m = \inf S \text{ and } M = \sup S \text{ i.e., } m \leq x \leq M \;\; \forall \;\; x \in S. \qquad \ldots(1)$$

We define a set of real numbers as follows :

$$T = \{x : x \text{ Exceeds only a finite number of elements of } S\}. \qquad \ldots(2)$$

From (2), it follows that

$$x \in T \Leftrightarrow \begin{cases} x \geq \text{only a finite number of elements of S} & \ldots(3) \\ \qquad \text{Or} \\ x \leq \text{infinite number of elements of S.} & \ldots(4) \end{cases}$$

($\because$ S is infinite)

From (1) and (4), we observe that $m \in T$ and so T is non-empty.

Again from (1), $M \geq x \ \forall \ x \in S$ (and S is infinite).

From (3), $\forall \ y \in T \Rightarrow y \geq$ only a finite number of elements of S.

Thus $M \geq y \ \forall \ y \in T$ and so M is an upper bound of T i.e., T is bounded above. Since T is a non-empty and bounded above subset of R, so by order-completeness property of R, T must have the supremum, say p i.e.,

$$\sup T = p.$$

We shall show that p is a limit point of S. Let $\varepsilon > 0$ be arbitrary.

Since $p + \varepsilon > p$ and p is an upper bound to T, $p + \varepsilon \notin T$

$\Rightarrow$ $p + \varepsilon \geq$ infinite number of elements of S, using (3)

$\Rightarrow$ infinite number of elements of $S \leq p + \varepsilon$. ...(5)

Since $p - \varepsilon < p$ and p is the l.u.b. of T, so $p - \varepsilon$ cannot be an upper bound of T. Consequently, there exists some $q \in T$ such that

$$q > p - \varepsilon, \ \text{ i.e., } \ p - \varepsilon < q, \ \text{ where } \ q \in T.$$

Now $q \in T \Rightarrow q \leq$ infinite number of elements of S, by (4).

Thus $p - \varepsilon < q \leq$ infinite number of element s of S. ...(6)

From (5) and (6), it follows that any nbd. $(p - \varepsilon, p + \varepsilon)$ of p contains infinite number of elements of S. Hence p must be a limit point of S.

Remark : The conditions of the **Bolzano-Weierstrass Theorem** cannot be relaxed.

We have seen in the corollary of Theorem that a finite set (which is of course bounded) can never have a limit point. Further there exist infinite unbounded sets having no limit point.

For Example,

(i) The set **Z** of integers is an infinite unbounded set, which has no limit point.

(ii) The set $S = \{1^2, 2^2, 3^2, \ldots n^2, \ldots\}$

is an infinite unbounded set, which has no limit point.

Remark : It may, however, be noticed that

$$S = \left\{1, 2, \frac{1}{2}, 3, \frac{1}{3}, 4, \frac{1}{4}, \ldots\right\}$$

is an infinite unbounded set, which has the limit point viz. 0.

Theorem 1 : Every finite set is bounded.

Proof : Let $A = \{a_1, a_2, \ldots, a_n\}$ be a finite set and

let $h = \min(a_1, a_2, \ldots, a_n)$ and $k = \max(a_1, a_2, \ldots, a_n)$

then $h \leq x \leq k, \ \forall \ x \in A$

Therefore A is bounded.

Theorem 2 : Every infinite bounded set of real numbers has a limit point.

Proof : Every infinite and bounded subset of **R** has a limit point i.e. atleast one limit point.

5. THE ARCHIMEDEAN PROPERTY OF REAL NUMBERS

If x, y are two positive real numbers, then there exists a positive integer n such that $ny > x$.

Proof : Suppose there does not exist any positive integer n such that $ny > x$. Then $ny \le x \ \forall \ n \in \mathbf{N}$. Consequently, the set

$$S = \{y, 2y, 3y, ... ny, (n + 1) y, ...\}$$

is non-empty and bounded above by x.

By order-completeness property, S has the supremum, say p.

$\therefore \quad (n + 1) y \le p \qquad \forall \ n \in N$

$\Rightarrow \quad ny \le p - y \qquad \forall \ n \in N, (p - y < p, \text{ as } y > 0)$

$\Rightarrow \quad p - y \ (< p)$ is an upper bound of S, which is a contradiction to the fact that $p = \sup S$.

Corollary 1 : If x be any positive real number, then there exists a positive integer n such that $n>x$.

Corollary 2 : Let x by any real number and y be a positive real number, then there exists a positive integer n such that $ny > x$.

Corollary 3 : For any real number x, there exists a positive integer n such that $n > x$.

Corollary 4 : For any real number x, there exists an integer m such that $m < x$.

Ex. If x be any positive real number, then there exists a positive integer n such that $\frac{1}{n} < x$.

Sol. Applying the Archimedean property, for $y = x > 0$ and $x = 1$, there exists a positive integer n such that $ny > x \Rightarrow nx > 1 \Rightarrow n > 1/x$. Hence $1/n < x$.

Remark : The integer n satisfying $n < x < n + 1$ is called an integral part of x and is denoted by [x].

Ex. If x and y are real numbers and $x > 1$, prove that there exists a positive n such that $x^n > y$.

Sol. Suppose that conclusion of the problem is false.

Then $x^n \le y \ \forall \ n \in N$. Consequently, the set

$$S = \{x, x^2, x^3, ..., x^n, ...\}$$

is non-empty subset of **R**, which is bounded above by y. By order-completeness property of **R**, S has the l.u.b. Let p = l.u.b. S. ...(1)

Since $\quad x > 1, px > p \Rightarrow p > p/x \Rightarrow p/x < p$

$\Rightarrow \quad p/x$ is not an upper bound of S, by (1) and so there exists some $x^n \in S$ such that $x^n > p/x \Rightarrow x^{n+1} > p$, where $x^{n+1} \in S$.

Thus shows that p is not an upper bound of S, which is a contradiction.

Hence $x^n > y$ for some positive integer n.

6. INTERVALS

The representation of real numbers as points on a straight line helps us to define four types of intervals.

Let a and b be any two real numbers such that $a < b$.

1. **Open Interval,** denoted by]a, b[, is defined as
   $(a, b) = \{x : a < x < b\}$.
   The end points a and b do not belong to (a, b).
2. **Closed Interval,** denoted by [a, b], is defined as
   $[a, b] = \{x : a \leq x \leq b\}$.
   The end points a and b belong to [a, b].

   Semi-Open or Semi-Closed Intervals are defined as follow :
3. $(a, b] = \{x : a < x \leq b\}$.
   Here $a \notin (a, b]$ and $b \in (a, b]$.
   The interval]a, b] is said to be open from the left and closed from the right.
4. $[a, b) = \{x : a \leq x < b\}$.
   Here $a \in [a, b)$ and $b \notin [a, b)$.

The interval [a, b) is said to be open from the right and closed from the left. The intervals (a, b] and [a, b) are also known as half-open or half-closed intervals.

7. OPEN AND CLOSED SETS

Open Sets

Definition : A set $S \subset \mathbf{R}$ is called an open set if it is a nbd of each of its points.

In Other Words : a set $S \subset \mathbf{R}$ is called an open set if for each $p \in S$, there exists some $\varepsilon > 0$ such that

$$p \in (p - \varepsilon, p + \varepsilon) \subset S \qquad \forall \qquad p \in S.$$

Examples

1. The set **R** of real numbers is an open set.
2. Every open interval (a, b) is an open set.
3. [a, b) is *not* an open set, since [a, b) is a *nbd.* of all is points *except* a.
   Similarly, (a, b], [a, b] are *not* open sets.
4. The set **Z** is integers is not an open set,
   since **Z** is not a *nbd.* of each of its points.
5. $G = (1, 2) \cup (3, 4)$ is an open set, as G is a *nbd.* of each of its points.
6. $\left(-\frac{1}{n}, \frac{1}{n}\right)$ is an open set for each $n \in \mathbf{N}$.
7. The empty set ϕ is an open set.
8. The singleton set {x} is **not** an open set, since for any $\varepsilon > 0$, $(x - \varepsilon, x + \varepsilon) \not\subset \{x\}$
9. The set **Q** of rational numbers is **not** an open set, since **Q** is not a nbd. of each of its points.
10. The set **R** – **Q** of all irrational numbers is **not** an open set.
11. The set $S = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right\}$ is **not** an open set, as S is not a nbd. of each of its points.

For Example, $(1 - \varepsilon, 1 + \varepsilon)$ is **not** an open set, as S is not a nbd. of each of its points.

For Example, $(1 - \varepsilon, 1 + \varepsilon)$ contains an infinite number of irrational numbers and so $(1 - \varepsilon, 1 + \varepsilon) \not\subset S$ for each $\varepsilon > 0$.

Theorem : The union of an arbitrary family of open sets is an open set.

Proof. Let $\{G_l : \lambda \in \Lambda\}$ be an arbitrary family of open set.

Let $\quad G = \bigcup_{\lambda \in \Lambda} G_\lambda$. We have to show that G is an open set.

Let x be any element of $G = \cup G_\lambda$.

Then $x \in G_\lambda$ for some $\lambda \in \Lambda$.

Since G_λ is given to be an open set and $x \in G_\lambda$, so G_λ is a nbd. of x. Thus there exist some $\varepsilon > 0$ such that

$$x \in (x - \varepsilon, x + \varepsilon) \subset G_\lambda$$

$$\Rightarrow \quad x \in (x - \varepsilon, x + \varepsilon) \subset G_\lambda \subset \bigcup_{\lambda \in \Lambda} G_\lambda = G$$

$$\Rightarrow \quad x \in (x - \varepsilon, x + \varepsilon) \subset G \ \forall \ x \in G$$

$\Rightarrow \quad$ G is a nbd. of x, $\forall\ x \in G$.

It follows that G is a nbd. of each of its points and consequently $G = G_\lambda$ is an open set.

Theorem : The intersection of a finite number of open sets is an open set.

Proof. First of all we show that the intersection of two open sets G_1 and G_2 is an open set.

Let x be any point of $G_1 \cap G_2$. Then $x \in G_1 \cap G_2$

$\Rightarrow$ $x \in G_1$ and $x \in G_2$

$\Rightarrow$ G_1 and G_2 are nbds. of x ($\because$ G_1 and G_2 are open sets)

$\Rightarrow$ $G_1 \cap G_2$ is nbd. of x

$\Rightarrow$ $G_1 \cap G_2$ is a nbd. of x for each $x \in G_1 \in G_2$.

Thus $G_1 \cap G_2$ is an open set.

Now we consider three open sets G_1, G_2, G_3. Then

$$G_1 \cap G_2 \cap G_3 = (G_1 \cap G_2) \cap G_3.$$

R.H.S. being the intersection of two open sets viz. $G_1 \cap G_2$ and G_3 is an open set and so $G_1 \cap G_2 \cap G_3$ is an open set.

Proceeding in a similar way, if $G_1, G_2, ..., G_n$ are a finite number of open sets, then

$G_1 \cap G_2 \cap ... \cap G_n$ is an open set.

Remark : The intersection of an arbitrary family of open sets may not be an open set.

Let us consider $G_n = (-1/n, 1/n)\ \forall\ n \in \mathbf{N}$. Then

$$G_1 = (-1, 1),\ G_2 = (-1/2, 1/2),\ G_3 = (-1/3, 1/3), ...$$

Since every open interval is an open set, therefore $\{G_n : n \in \mathbf{N}\}$ is a family of infinite number of open sets.

Clearly, $G_1 \cap G_2 = (-1, 1) \cap (-1/2, 1/2) = (-1/2, 1/2)$,

$G_1 \cap G_2 \cap G_3 = (-1/2, 1/2) \cap (-1/3, 1/3) = (-1/3, 1/3)$

and so on. Thus at every step, the size of the open interval is reducing and getting closer to 0.

Indeed $\bigcap_{n=1}^{\infty} G_n = \{0\}$, which is not an open set,

since for any $\varepsilon > 0$, $(0 - \varepsilon, 0 + \varepsilon) = (-\varepsilon, \varepsilon) \not\subset \{0\}$.

Example : Let $I_n = \left(x - \frac{1}{n}, x + \frac{1}{n}\right)$, n = 1, 2, ...

Prove that I_n is an open set for each positive integer n and $\bigcap_{n=1}^{\infty} I_n = \{x\}$.

Hint. Since each open interval is an open set, $(x - 1/n, x + 1/n)$ is an open set for n = 1, 2, 3, ... As shown above $\bigcap_{n=1}^{\infty}\left(-\frac{1}{n}, \frac{1}{n}\right) = \{0\}$.

Similarly,

$$\bigcap_{n=1}^{\infty}\left(x - \frac{1}{n}, x + \frac{1}{n}\right) = \{x + 0\} = \{x\}.$$

Example : Is every infinite set open? Justify your answer.

Hint. Every infinite set need not be an open set.

For Example,

(i) the set **Z** of integers is an infinite set, which is not an open set.

(ii) $S = \left\{1, \frac{1}{2}, \frac{1}{3}, ...\right\}$ is an infinite set, which is not an open set.

Definition : A set $S \subset \mathbf{R}$ is called a closed set if and only if its complement $\mathbf{R} \sim S$ is an open set.

S is closed $\Leftrightarrow$ $\mathbf{R} \sim S$ is open.

Examples

1. The set **R** of real numbers is a closed set, as $\mathbf{R} \sim \mathbf{R} = \phi$ is an open set.
2. The empty set ϕ is a closed set, as $\mathbf{R} \sim \phi = \mathbf{R}$ is open set.
3. The set **R** of real numbers is both open and closed. The empty set ϕ is both open and closed.
4. Every closed interval [a, b] is a closed set,

   $\because$ $\quad \mathbf{R} \sim [a, b] = (-\infty, a) \cup (b, +\infty)$

   and R.H.S. being the union of two open sets is an open set.
5. [a, b) is not a closed set,

   $\because$ $\quad \mathbf{R} \sim [a, b) = (-\infty, a) \cup [b, +\infty)$

   is not an open set as the R.H.S. is a nbd. of each of its points except b.

   Similarly, (a, b], (a, b) are not closed sets.
6. $[0, 3] \cap [1, 2]$ is a closed set, since $[0, 3] \cap [1, 2] = [1, 2]$, which is a closed set.
7. $F = [1, 2] \cup [2, 4]$ is a closed set, since $F = [1, 4]$ is a closed set.
8. Q is not a closed set, since $\mathbf{R} \sim \mathbf{Q}$ being the set of all irrational numbers is not an open set. Thus the set **Q** of rational numbers is neither a closed set nor an open set.

Theorem : (a) The union of a finite number of closed set is a closed set.

(b) The intersection of an arbitrary family of closed sets is a closed set.

(c) Given an Example to show that an arbitrary union of closed sets may not be a closed set.

Proof. (a) Let $F_1, F_2, ..., F_n$ be a finite number of closed sets so that $\mathbf{R} \sim F_1, \mathbf{R} \sim F_2, ..., \mathbf{R} \sim F_n$ are open sets.

$\therefore$ $\quad (\mathbf{R} \sim F_1) \cap (\mathbf{R} \sim F_2) \cap ... \cap (\mathbf{R} \sim F_n)$ is an open set.

By De Morgan's law,

$$\mathbf{R} \sim (F_1 \cup F_2 \cup ... \cup F_n) = (\mathbf{R} \sim F_1) \cap (\mathbf{R} \sim F_2) \cap ... \cap (\mathbf{R} \sim F_n)$$

$\Rightarrow$ $\quad \mathbf{R} \sim (F_1 \cup F_2 \cup ... \cup F_n)$ is an open set.

Hence $F_1 \cup F_2 \cup ... \cup F_n$ is a closed set.

(b) Let $\{F_\lambda : \lambda \in \Lambda\}$ be an arbitrary family of closed sets.

We have to prove that $F = \bigcap_{\lambda \in \Lambda} F_\lambda$ is a closed set.

Equivalently, we have to prove that $\mathbf{R} \sim F$ is an open set.

By De Morgan's law,

$$\mathbf{R} \sim F = \mathbf{R} \sim (\cap F_\lambda) = \cup (\mathbf{R} \sim F_\lambda).$$

Since each F_λ is a closed set, so $\mathbf{R} \sim F_\lambda$ is an open set.

Now $\cup (\mathbf{R} \sim F_\lambda)$ being an arbitrary union of open sets is an open set and so $\mathbf{R} \sim F$ is an open set. Hence $F = \cap F_\lambda$ is a closed set.

(c) Let $F_n = \left[\frac{1}{n}, 2\right] \ \forall\ n \in \mathbf{N}$.

Then $F_1 = [1, 2], F_2 = \left[\frac{1}{2}, 2\right], F_3 = \left[\frac{1}{3}, 2\right] \ldots$

Since every closed interval is a closed set, therefore $\{F_n : n \in \mathbf{N}\}$ is a family of infinite number of closed sets. We observe that

$$F_1 \cup F_2 = [1, 2] \cup \left[\frac{1}{2}, 2\right], F_3 = \left[\frac{1}{2}, 2\right],$$

$$F_1 \cup F_2 \cup F_3 = \left[\frac{1}{2}, 2\right] \cup \left[\frac{1}{3}, 2\right] = \left[\frac{1}{3}, 2\right] \text{ and so on.}$$

Indeed, $\bigcup_{n=1}^{\infty} F_n =]0, 2]$, which is not a closed set.

Example : Given an example of each of the following :

(a) an interval which is an open set.,

Ans. (a, b).

(b) an interval which is not an open set,

Ans. [a, b].

(c) an open set which is not an interval,

Ans. **R** or $(0, 1) \cup (2, 3)$.

(d) a set which is neither an interval nor an open set.

Ans. The set **Z** of integers.

Example : Give an example of each of the following :

(a) an interval which is a closed set,

Ans. [a, b].

(b) an interval which is not a closed set,

Ans. [a, b).

(c) a closed set which is not an interval,

Ans. **R** or $[0, 1] \cup [2, 3]$.

(d) a set which is neither an interval nor a closed set.

Ans. The set **Z** of integers.

(e) A set which is neither an open set nor a closed set.

Ans. The set **Q** of all rational numbers is neither an open set nor a closed set.

Example : Comment on the following statements :

(a) Can a finite set be open?

Ans. Yes. An empty set is a finite set, which is also an open set.

(b) Can a finite non-empty set be open?

Ans. No, Let S is a non-empty finite open set, then for any $x \in S$ and any $\varepsilon > 0$

$$(x - \varepsilon, x + \varepsilon) \not\subset S,$$

since the interval $(x - \varepsilon, x + \varepsilon)$ necessarily contains an infinite number of points and S is a finite set.

(c) Is every infinite set open?

Ans. No. The set **Z** of integers is an infinite set, which is not an open set.

(d) Is the intersection of any arbitrary family of open sets an open set?

Ans. No. $G_n = (-1/n, 1/n)\ \forall\ n \in \mathbf{N}$ is a family of infinite number of open sets and $\bigcap_{n=1}^{\infty} G_n = \{0\}$, which is not an open set,

Theorem : A set S is closed if and only if it contains all its limit points.

Equivalently, S is closed $\Leftrightarrow S' \subset S$.

Proof : By definition of a closed set,

S is closed $\Leftrightarrow \mathbf{R} \sim S$ is an open set

$\Leftrightarrow \mathbf{R} \sim S$ is a nbd. of each of its points

$\Leftrightarrow$ for each $x \in \mathbf{R} \sim S$, $\exists$ some $\varepsilon > 0$, such that

$(x-\varepsilon, x+\varepsilon) \subset \mathbf{R} \sim S$ or $(x-\varepsilon, x+\varepsilon) \cap S = \phi$

$(\because y \in (x-\varepsilon, x+\varepsilon) \Rightarrow y \in \mathbf{R} \sim S \Rightarrow y \notin S)$

$\Leftrightarrow$ for each $x \in \mathbf{R} \sim S$, $\exists$ a nbd. of x which contains no point of S

$\Leftrightarrow$ x is not a limit point of S $\forall\ x \in \mathbf{R} \sim S$.

$\Leftrightarrow x \notin S'\ \forall\ x \in \mathbf{R} \sim S$

$\Leftrightarrow x \notin S'\ \forall\ x \notin S$

$\Leftrightarrow S' \subset S$.

Remark. The above Theorem gives us a second definition of a closed set.

In order to show that a set S is closed, we must prove that if p is any limit point of S. then $p \in S$.

Corollary 1. Every finite set S is a closed set.

Proof. Since S is a finite set, S has not limit points. $\therefore S' = \phi$.

Since $\phi \subset S$. so $S' \subset S$. Hence S is a closed set.

Corollary 2. Every singleton is a closed set.

Proof. Any singleton set {x} is a finite set and hence by Corollary 1, it is a closed set.

Corollary 3. The set **Z** of integers is a closed set.

Proof. We know $\mathbf{Z}' = \phi$ and so $\mathbf{Z}' \subset \mathbf{Z}$. Hence **Z** must be a closed set.

Corollary 4. The set **N** of natural numbers is a closed set.

Proof. We know $\mathbf{N}' = \phi$ and so $\mathbf{N}' \subset \mathbf{N}$. Hence **N** is a closed set.

Theorem. Prove that the supremum of a non-empty bounded set is either the greatest member of the set or is a limit point of the set.

Or

Prove that the supremum of a non-empty bounded set S of real numbers when not a member of S must be a limit point of S.

Proof. Since S is non-empty and bounded above, S must have the supremum, say s

i.e., $s = \sup S$ (O-C Property).

$\Rightarrow \quad x \le s \quad \forall\ x \in S$.

If $s \le S$, then s is the greatest member of S. Let $s \notin S$.

We shall show that s is a limit point of S.

For any $\varepsilon > 0$, we have $s - \varepsilon < s$. ...(1)

Since s is the l.u.b. of S, so $s - \varepsilon$ cannot be an upper bound of S. Consequently, there exists some $t \in S$ such that

$$t < s < s + \varepsilon \Rightarrow t < s + \varepsilon. \qquad ...(2)$$

From (1) and (2), it follows that any nbd. $(s-\varepsilon, s+\varepsilon)$ of s contains a point $t \in S$, $t \neq s$. Hence s is a limit point of S.

Theorem : Show that a non-empty, bounded and closed set S contains its supremum and infimum.

Proof : Since S is non-empty and bounded above S must have a supremum say s

i.e., s = sup S (O-C property). If we show that $s \in S$, then S has a maximum viz. s. We now proceed to show that s is a limit point of S.

Let $\varepsilon > 0$ be arbitrary. Since s = l.u.b.S. $s - \varepsilon$ is not an upper bound of S and so there exists some $t \in S$ such that

$$t > s - \varepsilon \quad \text{or} \quad s - \varepsilon < t \qquad ...(1)$$

Clearly, $$t < s < s + \varepsilon \qquad ...(2)$$

From (1) and (2), it follows that $(s - \varepsilon, s + \varepsilon)$ contains a point $t \in S$, $t \neq s$ for all $\varepsilon > 0$.

Thus s is a limit point of S. Since S is a closed set, $s \in S$.

Hence s is a maximum of S. Similarly, s' = inf S is a minimum of S.

Theorem : Prove that the derived set of any set is a closed set.

Proof : Let S' be the derived set of a set S. We have to show that S' is a closed set.

Equivalently, we have to show that if p is limit point of S', then $p \in S'$.

Suppose p is a limit point of S'. The for any $\varepsilon > 0$, $(p - \varepsilon, p + \varepsilon)$ contains a point q of S'. $q \neq p$. Now $q \in S' \Rightarrow q$ is a limit point of S.

$\Rightarrow$ Every nbd. of q (in particular, $(p - \varepsilon, p + \varepsilon)$) contains a point r of S, $r \neq q$

$\Rightarrow$ for each $\varepsilon > 0$, $(p - \varepsilon, p + \varepsilon)$ contains a point $r \in S$, where $r \neq p$

($\because$ $r \neq q$ and $q \neq p$)

$\Rightarrow$ p is a limit point os $S \Rightarrow p \in S'$. Hence S' is closed set.

Theorem : The derived set of a bounded set is bounded.

Proof : Let S be any bounded set whose bounds are

$$m = \inf S \quad \text{and} \quad M = \sup S$$

i.e., $$m \le x \le M \quad \forall \quad x \in S \qquad ...(1)$$

We shall prove that

$$m \le x \le M \quad \forall \quad x \in S'$$

Let, if possible, there exist some $y \in S'$ such that $y > M$.

Now, $y > M \quad \Rightarrow \quad y - M > 0$.

We choose $\varepsilon > 0$ such that

$$\varepsilon < y - M \quad \Rightarrow \quad M < y - \varepsilon \qquad ...(2)$$

From (1) and (2),

$$x \le M < y - \varepsilon \ \forall \quad x \in S.$$

In follows that there exists a nbd. $(y - \varepsilon, y + \varepsilon)$ of y which does not contain any point of S and so y in not a limit point of S i.e., $y \notin S'$, which is contradiction.

Thus $x \le M \quad \forall \quad x \in S'$.

Similarly, we can show that

$$m \le x \quad \forall \quad x \in S'$$

Hence $m \le x \le M \quad \forall \quad x \in S'$ and so S' is bounded.

Ex. Show that $S = \left\{1, -1, 1\frac{1}{2}, -1\frac{1}{2}, 1\frac{1}{3}, -1\frac{1}{3}, ...\right\}$ is closed but not open.

Sol. The only limit points of the set S are 1 and – 1 i.e., S' = {1, – 1}.

Since $S' \subset S$, S is a closed set.

Since S is not a nbd. of each of its points, S is not an open set.

Notice the for each $\varepsilon > 0$, $1 \in (1 - \varepsilon, 1 + \varepsilon) \not\subset S$.

Example : Comment on the following statements :

(a) Can a singleton be closed?

Ans. Yes. Every singleton is closed

(b) Can a finite set be closed?

Ans. Yes. Every finite set is closed

(c) Is every finite set closed?

Ans. Yes.

(d) Is every infinite set closed?

Ans. No. $S = \{1, \frac{1}{2}, \frac{1}{3}, ...\}$ is an infinite set which is not closed, since its only limit point 0 does not belong to the set S.

(e) Is the union of an arbitrary family of closed sets a closed set?

Ans. No. $\{F_n = [1/n, 2] : n \in N\}$ is a family of infinite number of closed sets, and $\bigcap_{n=1}^{\infty} F_n = (0, 2]$, which is not a closed set.

(f) Is the intersection of an arbitrary family of closed sets a closed set?

Ans. Yes.

Ex. Give Examples of sets S and S' such that

(a) $S \cap S' = \phi$, (b) $S \subset S'$, (c) $S' \subset S$, (d) $S = S'$.

Sol. (a) Let $S = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ...\}$. Then $S' = \{0\}$. Clearly, $S \cap S' = \phi$.

(b) Let $S = (0, 1)$ so that $S' = [0, 1]$. Obviously, $S \subset S'$.

(c) Let $S = \{1, -1, 1\frac{1}{2}, -1\frac{1}{2}, 1\frac{1}{3}, ...\}$. Then $S' = \{1, -1\} \subset S$.

(d) Let $S = [0,1]$ so that $S' = [0,1]$. We have $S = S'$.

Ex. Find the limit points of the following :

(i) **N**, (ii) (a, b], (iii) **R** ~ **Q**, (iv) finite set.

Sol. (i) **N** has no limit point.

(ii) $\{(a, b]\}' = [a,b]$. Hence each point of [a, b] is a limit point of (a, b].

(iii) **R** ~ **Q**, consists of all irrational numbers. Every real number $p \in$ **R** is a limit point of **R** ~ **Q**, since for each $\varepsilon > 0$, $(p - \varepsilon, p + \varepsilon)$ contains an irrational number other than p.

(iv) A finite set has no limit point .

Example : State whether the following subsets of **R** are open, closed :

(i) the set **Z** of all integers.

Ans. **Z** is closed. ($\because$ $\mathbf{Z}' = \phi \subset \mathbf{Z}$.)

However, **Z** is not open as **Z** is not a nbd. of any of its points.

(ii) $S = \left\{\frac{1}{n} : n \in \mathbf{N}\right\}$.

Ans. S is neither closed nor open.

We have $S' = \{0\} \not\subset S$. Also S is not a nbd. of any of its points.

(iii) The segment (a, b).

Ans. It is an open set, since it is an open interval.

However, it is not a closed set, since $\{(a, b)\}' = [a, b] \not\subset (a, b)$.

(iv) The set **N** of natural numbers.

Ans. **N** is closed. $(\because \mathbf{N}' = \phi \subset \mathbf{N})$

However, **N** is not open as **N** is not a nbd. of any of its points.

(v) $S = \{1, 2, 3, 4\}$.

Ans. S is a closed set, since it is a finite set and every finite set is closed.

However, S is not open, since S is not a nbd. of any of its points.

(vi) The set **R** of real numbers.

Ans. **R** is both open and closed.

(vii) $S = (0, 1) \cup (2, 3)$.

Ans. S being the union of two open sets is an open set.

Now $S' = \{(0, 1)\}' \cup \{(2, 3)\}' = [0, 1] \cup [2, 3] \not\subset S$.

Hence. S is not a closed set.

(viii) $S = \left\{1, -1, \frac{1}{2}, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{3}, ...\right\}$.

Ans. $S' = \{0\} \not\subset S$ and so S is not closed.

Since S is not a nbd. of each of its points, S is not open.

(ix) The set **Q** as rational numbers.

Ans. We know $\mathbf{Q}' = \mathbf{R} \not\subset \mathbf{Q}$ and so **Q** is not closed.

Further **Q** is not a nbd. of any of its points and so **Q** is not open.

(x) $S = [0, 1] \cup [2, 3]$.

Ans. S being the union of two closed sets is a closed set (Every closed interval is a closed set).

Since S is not a nbd. of the points 0, 1, 2, and 3, S is not an open set.

Ex. It F is a closed bounded set, then every infinite subset S of F has limit point in F.

Sol. Since S is a subset of the bounded set F, so S is a bounded and infinite set. By **Bolzano Weierstrass Theorem,** S has a limit point, say p. Since $S \subset F$, so p is also a limit point of F. Since F is a closed set and p is a limit point of F, so $p \in F$. Hence S has its limit point in F.

Ex. If A is open and B is closed, show that $A \sim B$ is open and $B \sim A$ is closed.

Sol. Since B is closed, $\mathbf{R} \sim B$ is an open set. Now $A \sim B = A \cap (\mathbf{R} \sim B)$, which being the intersection of two open sets is an open set. Thus $A \sim B$ is an open set.

Again $B \sim A = B \cap (\mathbf{R} \sim A)$, which being the intersection of two closed sets is a closed set. Thus $B \sim A$ is a closed set.

8. INTERIOR AND EXTERIOR OF A SET

Interior Points of A Set

A point x is an interior point of a set S is S is a nbd of x. In other words, x is an interior point of S if $\exists$ an open interval (a, b) containing x and contained in S, i.e., $x \in (a, b) \subseteq S$.

Thus a set is a neighbourhood of each of its interior points.

Interior of A Set : The set of all interior points of set is called the interior of the set. The interior of a set S is generally denoted by S^i or into S.

Example : The interior of the set **N** or 1 or **Q** is the null set, but interior of **R** is **R**.

Example : The interior of a set S is a subset of S, i.e., $S^i \subseteq S$.

Interior, Exterior and Boundary of A Set

Definition :

(i) Let A be a subset of **R** and let $p \in A$. Then p is called an interior point of A is there exists an ε-nbd of p contained in A, i.e., if there exists an $\varepsilon > 0$ such that $]\, p - \varepsilon, p - \varepsilon\, [\subset A$. The set of all interior points of A is called the interior of A and is denoted by A^o or by int A.

(ii) A point p is called an Exterior point of A if there exists an ε-nbd of p contained in the complement A' of A. The set of all Exterior points of A is called Exterior of A and is denoted by Ext A.

(iii) A point p is called a boundary point (or frontier point) of a if it is neither an interior nor an Exterior nor an Exterior point of A. The set of all boundary points of A called the boundary (or frontier) of A and is denoted by b (A) [or Fr(A)].

Result :

(i) A^o is the set of all those points of A which are not the limit points of A.'

(ii) A point p is an Exterior point of if and p is not an adherent point of A.

(iii) A point p is a boundary point of A if and only if every ε-nbd of p intersects both A and A'.

(iv) A^o is an open set,

(v) A^o is the largest open set contained in A

(vi) A is open if and only if $A^o = A$.

Theorem : Let A, B be any subsets of **R** Then

(i) $A^o \subset A$, (ii) $A \subset B \Rightarrow A^o \subset B^o$

(iii) $(A \cap B)^o = A^o \cap B^o$ (iv) $A^o \cup B^o \subset (A \cup B)^o$

(v) $A^{oo} = A^o$

Proof : We prove only (iii). The proofs of others are easy and let as Exercise for the reader. We have by (ii)

$$A \cap B \subset A \Rightarrow (A \cap B)^o \subset A^o \text{ and } A \cap B \subset B \Rightarrow (A \cap B)^o \subset B.$$

$\therefore$ $(A \cap B)^o \subset A^o \cap B^o$. ...(1)

Again $A^o \subset A$ and $B^o \subset B \Rightarrow A^o \cap B^o \subset A \cap B$.

Also $A^o \cap B^o$ is open, being the intersection of two open sets.

Thus $A^o \cap B^o$ is an open set contained in $A \cap B$. But $(A \cap B)^o$ is the largest open set contained in $A \cap B$. Hence

$A^o \cap B^o \subset (A \cap B)^o$ ...(2)

From (1) and (2), $(A \cap B)^o = A^o \cap B^o$

9. COMPACTNESS

(I) Open Cover of A Set : Definition : Let $S \subset \mathbf{R}$ and $C = \{A_i \mid i \in I\}$, **R** be the group of open set in **R**. Then C is called the open cover of the set S, if

$$S \subset \bigcup_{i \in I} A_i$$

i.e., every element of S exist atleast is one of the open set of C.

(II) Finite Subcover : Definition : Any finite subcover $C' = \{A_j \mid j \in J \subset I\}$, S of open subcover $C = \{A_i \mid i \in I\}$ of any set S is called a finite subcover of S, if

$$S \subset \bigcup_{j \in J} A_j$$

Example : Group $C = \{(-n, n) \mid n \in \mathbf{N}\}$ and $C' = \{(-3n, 3n) \mid n \in \mathbf{N}\}$ are two open covers of **R** and C' is an open cover of C.

Example : Group of open intervals $\left\{\left(\frac{x}{2}, \frac{3x}{2}\right) \mid x \in (0, 1)\right\}$ is the open cover of the set (0, 1).

Compact Set : Definition

A non-empty subset of **R** is said to be compact if a finite subcover of S is contained in each open cover of any set S.

Example : A finite subset $S = \{x_1, x_2, ...x_n\}$ of **R** is a compact set, since if $C = \{A_i \mid i \in I\}$ is an open cover of S, then for every $x_r \in S$, there exist an open A_r in C, such that $x_r \in A_r$, r = 1, 2, ...n.

Now if $C' = \{A_r \mid r = 1, 2, ...n\}$, then $S \subset \bigcup_{r=1}^{n} A_r$

Therefore C' is a finite cover of the set S.

Consequently, S is compact set.

Heine Borel (H-B) Theorem : Statement

Every open cover of a closed and bounded subset S of real number contains a finite subcover of S.

Proof. Let S be a closed and bounded subset of the set **R** of real numbers.

Now since S is bounded, therefore there exists an closed interval

$I_0 = [a, b]$ such that $S \subset [a, b]$

where a and b are the infimum and supremum respectively of S.

Let $C = \{A_i \mid i \in N\}$ be an open cover of [a, b].

Hence C is also an open cover of S.

Therefore $S \subset [a, b] \subset \bigcup_{i \in N} A_i$

If possible, suppose no finite subcover of [a, b] is contained in C i.e., any finite sub group of C is not the subcover of $I_0 = [a, b]$.

Then bisecting I_0 at $\frac{1}{2}(a + b)$, there will not exist finite subcover in C of atleast one of the intervals

$\left[a, \frac{1}{2}(a+b)\right]$ and $\left[\frac{1}{2}(a+b), b\right]$.

Denote this interval by $I_2 = [a_2, b_2]$.

Repeating this process, we get a sequence of nested intervals

$$I_0 \supset I_1 \supset I_2 \supset \ldots \supset I_n \supset \ldots$$

in which no finite cover of any interval exist in C.

Clearly, Length of $I_n = \|I_n\| = b_n - a_n = \dfrac{b-a}{2^n}$

$\Rightarrow \quad \lim\limits_{n\to\infty} |I_n| = 0$

Therefore by the property of nested intervals, there exist only one point say x_0 in $\bigcap\limits_{n=1}^{\infty} I_n$. This point x_0 exists in each of the above nested interval, therefore $x_0 \in [a, b]$. But C is the open cover of [a, b]. Therefore for x_0, there exists an open set say A_0 in C such that $x_0 \in A_0$. Now since A_0 is an open set, therefore $x_0 \in A_0 \Rightarrow \exists \varepsilon > 0$ s.t. $(x_0 - \varepsilon, x_0 + \varepsilon) \subset A_0$.

Again, since $\lim \|I_n\| = 0$, therefore a natural number k may be chosen such that $\dfrac{b-a}{2^k} < \varepsilon$, then $I_k \subset (x_0 - \varepsilon, x_0 + \varepsilon) \subset A_0$

Consequently, there will be a finite cover of one element of the interval I_k in C. This is contrary to the fundamental property of the above intervals $I_0, I_1, I_2, \ldots$.

Thus our assumption is false.

Therefore there exist a finite cover of [a, b] in C.

Hence finite cover of S is also in C.

The above theorem can also be stated as follows:

Alternative definition of Compact set

A non-empty subset of **R** is said to be compact if it is closed and bounded.

Properties of Compact Sets

Theorem : Every compact subset of real numbers is closed and bounded. A subset of real numbers is compact iff it is closed and bounded.

Proof. S is bounded :

Let S be the compact subset of **R** and $A_n = (-n, n)$, then

$$C = \{A_n \mid n \in \mathbf{N}\}$$

is an open cover of **R**. Hence this is also an open cover of S.

Since S is a compact set, therefore there exist finite natural numbers $n_1, n_2, \ldots, n_k$ such that finite subgroup $\{A_{n_1}, A_{n_2}, \ldots, A_{n_k}\}$ of C will also be subcover of S. Let $n_0 = \max\{n_1, n_2, \ldots, n_k\}$, then clearly,

$$S \subset A_{n_0} = (-n_0, n_0)$$

Consequently, S is a bounded set.

S is closed :

For this we shall show that no point of **R** – S is limit point of S i.e. all the limit points of S exist in S itself.

Therefore let $a \in \mathbf{R} - S$, then $a \notin S$. For every $n \in \mathbf{N}$,

$B_n = \left[a - \dfrac{1}{n}, a + \dfrac{1}{n}\right]$ is a closed set. $\Rightarrow \mathbf{R} - B_n$ is an open set.

$\Rightarrow \quad C' = \{\mathbf{R} - B_n \mid n \in \mathbf{N}\}$ is a group of open set.

Again, clearly $\bigcap_{n=1}^{\infty} B_n = \{a\}$. Now since $a \notin S$, therefore

$$S \subset [\mathbf{R} - \cap B_n : n \subset \mathbf{N}] = \cup [\mathbf{R} - B_n : n \in \mathbf{N}] \qquad \text{[By Demorgan's law]}$$

$\Rightarrow$ C' is a cover of S.

But S is a compact set, therefore by the compactness property, there will exist finite natural numbers $m_1, m_2, ..., m_s$ such that each point of S will exist atleast in one of

$$\mathbf{R} - B_{m_1}, \mathbf{R} - B_{m_2}, ..., \mathbf{R} - B_{m_s}$$

Therefore if $x \in S$, then

$$x \in S \Rightarrow \exists\ i \in \{1, 2, 3, ..., r\} \text{ r.t. } x \in \mathbf{R} - B_{m_i}$$

$\Rightarrow$ any point of S does not exist in $B_{m_i} = \left[a - \frac{1}{m_i}, a + \frac{1}{m_i}\right]$

$\Rightarrow$ a is not the limit point of S $\Rightarrow$ all the limit points of S exist in S.

$\Rightarrow$ S is closed.

Theorem. **R** is not compact.

Solution. Let $A_n = (-n, n)$, then A_n is an open set for each value of natural number and $C = \{A_n \mid n \in \mathbf{N}\}$ is the open cover of **R**,

because $\quad \mathbf{R} \subset \bigcup_{n \in N} A_n$

If $\quad C' = \{A_{n_1}, A_{n_2}, ..., A_{n_k}\}$

be the finite subgroup of the cover C, then let

$$n_0 = \max \{n_1, n_2, ..., n_k\}$$

then $\quad n_0 \notin A_{n'} \qquad \forall\ i = 1, 2, ..., k.$

Therefore clearly, any finite subset of C can not be the cover of **R**.

Consequently, **R** is not compact.

Theorem : No open interval (a, b) is compact.

Solution. Let $A_n = \left(a + \frac{1}{n}, b\right)$, then for every natural number n, A_n is an open set and

$C = \{A_n \mid n \in \mathbf{N}\}$ is an open cover of (a, b) since $(a, b) \subset \bigcup_{n \in N} A_n$.

Now we will show that no finite subgroup of C can not be the subcover (a, b). If

$$C' = \{A_{n_1}, A_{n_2}, ..., A_{n_k}\}$$

be the finite subgroup of the cover C, then let $n_0 = \max \{n_1, n_2, ..., n_k\}$ then clearly, $\left(a, a + \frac{1}{n_0}\right) \not\subset \bigcup_{i=1}^{k} A_{n_i}$

Therefore C' can not be the subcover of (a, b).

Consequently, (a, b) is not compact.

Cantor Ternary Set : [Γ]

I Stage : Denote the closed interval [0, 1] by F_0.

Divide the interval into three equal parts and remove the middle one third open interval $I_{1,1} = \left]\frac{1}{3}, \frac{2}{3}\right[$.

Remaining two disjoint open intervals $J_{1,1} = \left[0, \frac{1}{3}\right]$ and $J_{1,2} = \left[\frac{2}{3}, 1\right]$ are of length $= \frac{1}{3}$.

Now $\quad J_{1,1} \cup J_{1,2} = F_1$ (say) ...(1)

II Stage : Now again divide the above two closed intervals $J_{1,1}$ and $J_{1,2}$ into three equal parts and remove the middle one third open intervals of each of $I_{2,1} = \left]\frac{1}{9}, \frac{2}{9}\right[$ and $I_{2,2} = \left]\frac{7}{9}, \frac{8}{9}\right[$.

Thus there remains the following four (2^2) disjoint closed intervals :

$$J_{2,1} = \left[0, \frac{1}{9}\right],\ J_{2,2} = \left[\frac{2}{9}, \frac{1}{3}\right],\ J_{2,3} = \left[\frac{2}{3}, \frac{7}{9}\right],\ J_{2,4} = \left[\frac{8}{9}, 1\right]$$

The length of each of the above interval $= \frac{1}{9}$

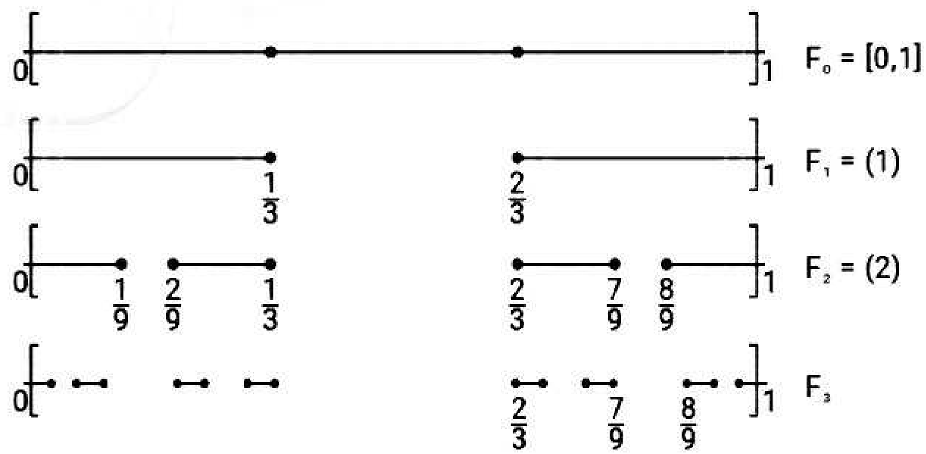
Now $\quad J_{2,1} \cup J_{2,2} \cup J_{2,3} \cup J_{2,4} = F_2$ (say) ...(2)

and $\quad O_2 = I_{2,1} \cup I_{2,2}$

n^{th} Stage :

In the nth stage (2^{n-1}) open intervals $I_{n,1}, I_{n,2}, \ldots, I_{n,2^n-1}$ will be removed and there remains 2^n closed intervals $I_{n,1}, I_{n,2}, \ldots, I_{n,2^n}$ (left to right) where length of each subinterval $= \left(\frac{1}{3}\right)^n$

. . . .



Now $\bigcup_{k=1}^{2^n} J_{n,k} = F_n$ and $\bigcup_{k=1}^{2^{n-1}} I_{n,k} = O_n$ (say) $\quad (n \in N)$

Now the set $\bigcap_{n=1}^{\infty} F_n = [0,1] \cap \left[\bigcup_{n=1}^{\infty} O_n\right]$ is called Cantor's theory sat and is denoted by Γ'.

Remark. The set Γ is non-empty because the end points of the closed intervals have not been removed and are in the set Γ.

10. CONNECTEDNESS

Connected Set. Definition : A subset S of the set of real numbers **R** is called connected set if no two disjoint open sets A and B of **R** exist such that

$$S \subset A \cup B \text{ and } A \cap S \neq \phi,\ B \cap S \neq \phi$$

The set which is not connected, is known as Disconnected Set.

Example 1 : $S = \{x \in \mathbf{R} \mid 0 \leq x \leq 1\}$ is a connected set because no two disjoint subsets A and B of **R** such that

$$S \subset A \cup B \text{ and } A \cap S \neq \phi,\ B \cap S \neq \phi$$

Example 2 : Each open interval (a, b) is a connected set.

Theorem : Any subset S of real numbers is a connected set iff S is an interval.

Proof. S is connected $\Rightarrow$ S is an interval :

If possible, let S be not an interval, then there will exist three real numbers a, q, b such that $a < q < b$ and $a \in S$. $b \in S$ but $q \notin S$.

Clearly, $a \in (-\infty, q) = A$ (say), $b \in (q, \infty) = B$ (say)

Therefore there exist non-empty open sets A and B such that

$$A \cap B = \phi,\ S \subset A \cup B,\ S \cap A, \neq \phi.\ S \cap B = \phi$$

Consequently, S is not a connected set.

Therefore, if S is not an interval, then this will not be a connected set.

Conversely : Set S is an interval $\Rightarrow$ S is connected

Let S be an interval and A, B be two open sets such that

$$A \cap S \neq \phi,\ B \cap S \neq \phi \text{ and } A \cap B = \phi.$$

Let $a \in A \cap S$ and $b \in S \cap S$. since $a \neq b$,

Therefore, let $a < b$ then $[a, b]$ will be a subinterval of the set S.

Let u be the supremum of the set $A \cap [a, b]$.

Since a, b are internal points of A and B respectively

and $A \cap B = \phi$, therefore $a < u < b$

Now if $\varepsilon > 0$ is a number such that $a < u - \varepsilon < u + \varepsilon < b$

then there will be atleast one point of A in $(u - \varepsilon, u)$

and no point of A will be there in $(u, u + \varepsilon)$.

Consequently, $(u - \varepsilon, u + \varepsilon)$ is not the subset of the sets A or B.

Again, since A and B are open, therefore $u \notin A$ and $u \notin B$.

$\therefore$ $S \not\subset A \cup B$

$\Rightarrow$ S is a connected set.

Therefore S is connected $\Rightarrow$ S is an interval.

Theorem : The set **R** of real numbers is a connected set.

Proof. If possible, let **R** be not connected set then there exist two disjoint open sets A and B such that

$$\mathbf{R} \subset A \cup B \text{ and } A \cap \mathbf{R} \neq \phi,\ B \cap \mathbf{R} \neq \phi$$

Let $a \in A$, $b \in B$ and

$$A_1 = \{x \in [0, 1] : a + x(b - a) \in A\}$$

and $$B_1 = \{x \in [0, 1] : a + x(b - a) \in B\}$$

Now since $[a, b] = \{\alpha = a + s(b - a) : 0 \leq x \leq 1\} \subset \mathbf{R} \subset A \cup B$

$\therefore \quad A_1 \cup B_1 = [0, 1]$

Again since $a \in A$ and $a = a + 0(b - a) \Rightarrow 0 \in A_1 \Rightarrow A_1 \neq \phi$

Similarly, $b \in B$ and $b = a + 1(b - a) \Rightarrow 1 \in B_1 \Rightarrow B_1 \neq \phi$

Since A, B are two disjoint open sets, therefore A_1 and B_1 are also two disjoint open sets.

Consequently $\{A_1, B_1\}$ shows the disconnectivity of the interval [0, 1] which contradicts the statement of the theorem proved earlier.

Therefore our earlier assumption is false i.e., **R** is a connected set.

Ex. That the only subsets of **R** which are both open and closed are **R** and ϕ.

Sol. Let A be a subset of **R** which is both closed as well as open.

Then $\sim A = \mathbf{R} - A = B$ (say) will also be open as well as closed.

If $A \neq \phi$ and $A \neq \mathbf{R}$, then there exist two non empty disjoint subsets A and B such that

$$\mathbf{R} \subset A \cup B,\ A \cap \mathbf{R} \neq \phi,\ B \cap \mathbf{R} \neq \phi$$

Consequently, **R** will not be connected set which contradicts the above theorem.

Therefore $A = \phi$ or **R**.

i.e., ϕ and **R** are the only two subsets of **R** which are both open and closed.

11. POWER SERIES

Definition 1 : A series of the form $\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \ldots\ldots + a_n x^n + \ldots\ldots.$

where $a_0, a_1, a_2, \ldots\ldots$ are constant, is called a power series in x.

Definition 2 : A constant $R \geq 0$ such that $\Sigma a_n (x - x_0)^n$ converges for $|x - x_0| < R$ and diverges for $|x - x_0| > R$ is called the radius of convergence of the series.

Interval (domain) of Convergence

The interval $|x| < R$ or $-R < x < R$ is called the interval (domain) of convergence.

Examples

1. $\sum \frac{x^n}{n^2}$ converges when $|x| \leq 1$ and diverges when $|x| > 1$.

Radius of Convergence

The radius of convergence is a number R such that the series

$\Sigma a_n (x - x_0)^n$

converges absolutely for $|x - x_0| < R$ and diverges for $|x - x_0| > R$

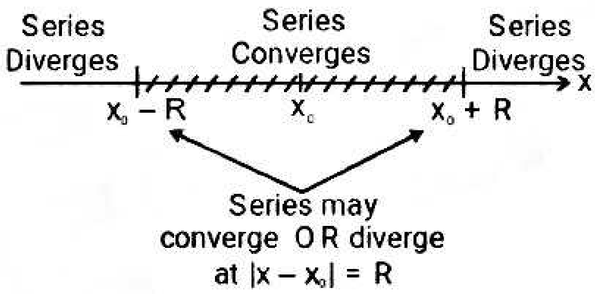


Fig. : Radius of Convergence

Note that :

- If the series converges ONLY at $x = x_0$, $R = 0$.
- If the series converges for ALL values of x, R is said to be infinite

Formula for The Radius of Convergence

If the power series $\Sigma a_n x^n$ is such that $a_n \neq 0$ for all n and $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \frac{1}{R}$, then $\Sigma a_n x^n$ is convergent (absolutely convergent) for $|x| < R$ and divergent for $|x| > R$.

OR

The radius of convergence R of a power series $\Sigma a_n x^n$ may be determined by the following

$$R = \lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right| = 1 \div \lim_{n \to \infty} \sup |a_n|^{1/n}$$

$R = \infty$, when $\lim |a_n|^{1/n} = 0$

$R = 0$, when $\lim |a_n|^{1/n} = \infty$

Ex. Find the radius of convergence of $\sum_{n=1}^{\infty} nx^{n-1}$.

Sol. Here $\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\frac{n}{n+1} = 1$.

Hence the radius of convergence = 1.

Ex. Find the radius of convergence of the series

$$x + \frac{1!}{2^2}x^2 + \frac{2!}{3^3}x^3 + \frac{3!}{4^4}x^4 + \ldots$$

Sol. $a_n = \frac{(n-1)!}{n^n}$, $a_{n+1} = \frac{n!}{(n+1)^{n+1}}$

$$\therefore \quad R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\frac{(n-1)!}{n!}\frac{(n+1)^{n+1}}{n^n}$$

$$= \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^{n+1} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n\left(1+\frac{1}{n}\right) = e$$

Hence the radius of convergence = e.

Ex. Find the radius of convergence and the exact interval of convergence of the following power series :

(i) $\sum \frac{(n+1)}{(n+2)(n+3)}x^n$, (ii) $\sum \frac{(n!)^2 x^{2n}}{(2n)!}$

Sol. **(I)** $a_n = \frac{(n+1)}{(n+2)(n+3)}$, $a_{n+1} = \frac{(n+2)}{(n+3)(n+4)}$.

$$\therefore \quad R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \frac{(n+1)(n+4)}{(n+2)^2} = 1.$$

Thus the given series converges for | x | < 1 and diverges for | x | > 1. We now check at the end points x = 1, –1.

For x = 1, the nth term of the series is

$$u_n = \frac{(n+1)}{(n+2)(n+3)} \sim \frac{n}{n^2} = \frac{1}{n}, \text{ which diverges.}$$

However for x = –1, the given series is an alternating series for which $u_{n+1} < u_n \ \forall$ n and $\lim_{n\to\infty} u_n = 0$. Thus, by Leibnitz's test, the series converges for x = –1. Hence the interval of convergence is [–1, 1[.

(ii) Given $\sum \frac{(n!)^2 x^{2n}}{(2n)!}$

Let $x^2 = y$

then power series reduces to

$$\sum \frac{(n!)^2 y^n}{(2n)!}$$

We have $a_n = \frac{(n!)^2}{(2n)!}$, $a_{n+1} = \frac{[(n+1)]^2}{(2n+2)!}$

$$\frac{a_n}{a_{n+1}} = \frac{\frac{(n!)^2}{(2n)!}}{\frac{((n+1)!)^2}{((2(n+1))!)}}$$

$$\frac{a_n}{a_{n+1}} = \frac{(n!)^2}{2n!} \times \frac{(2n+2)!}{((n+1)!)^2}$$

$$= \frac{(n!)^2}{2n!} \times \frac{(2n+2)\times(2n+1)\times(2n)!}{(n+1)^2(n!)^2}$$

$$= \frac{(2n+2)\times(2n+1)}{(n+1)^2}$$

$$R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\frac{(2n+2)(2n+1)}{(n+1)^2}$$

$$= \lim_{n\to\infty}\frac{n\left(2+\frac{2}{n}\right).n\left(2+\frac{1}{n}\right)}{n^2\left(1+\frac{1}{n}\right)^2}$$

$$= \lim_{n\to\infty}\frac{\left(2+\frac{2}{n}\right)\left(2+\frac{1}{n}\right)}{\left(1+\frac{1}{n}\right)^2}$$

$$= 4$$

So the power series converges for $|y| < 4$ i.e. $|x|^2 < 4 \Rightarrow |x| < 2$ and diverges for $|y| > 4$. i.e. $|x|^2 > 4$. So the interval of convergence is $|x| < 2$ i.e. $(-2, 2)$. And Radius of convergence is 2.

Ex. Prove that the power series

$$1 + \frac{a.b}{1.c}x + \frac{a(a+1)\,b(b+1)}{1.2c\,(c+1)}x^2 + \ldots$$

has unit radius of convergence.

Sol. $a_n = \dfrac{a(a+1)\,...\,(a+n-1)\,b(b+1)\,...\,(b+n-1)}{1.2...\,n\,c(c+1)...(c+n-1)}$ (omitting the first term)

$$a_{n+1} = \frac{a(a+1)...(a+n)\,(b+1)\,...\,(b+n)}{1.2....(n+1)\,c(c+1).....(c+n)}$$

$$\therefore \quad R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\frac{(n+1)(c+n)}{(a+n)(b+n)}$$

$$= \lim_{n\to\infty}\frac{(1+1/n)(1+c/n)}{(1+a/n)(1+b/n)} = 1.$$

Ex. Find the domain of convergence of the series

(i) $\sum \dfrac{x^n}{n^3}$, (ii) $\sum \dfrac{(-1)^n\,(x-1)^n}{2^n(3n-1)}$

Sol. **(i)** $a_n = \dfrac{1}{n^3}$, $a_{n+1} = \dfrac{1}{(n+1)^3}$ so that

$$\lim_{n\to\infty}\frac{a_n}{a_{n+1}} = \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^3 = 1. \text{ Thus } R = 1.$$

For $x = 1$, the given power series is $\sum \dfrac{1}{n^3}$, which is convergent.

For $x = -1$, the given power series is

$$-1 + \frac{1}{2^3} - \frac{1}{3^3} + \frac{1}{4^3} - ....$$

which is convergent, by Leibnitz's Test.

Hence the domain for convergence is $[-1, 1]$.

(ii) The given power series is about the point $x = 1$. We have

$$\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\frac{2^{n+1}(3n+2)}{2^n(3n-1)} = 2. \text{ Thus } R = 2.$$

The domain of convergence of the given power series is

$$-2 + 1 < x < 2 + 1 \text{ or } -1 < x < 3.$$

For $x = 3$, given power series is $\sum \dfrac{(-1)^n}{3n-1}$, which is convergent by Leibnitz's Test.

Hence the domain of convergence of the given power series is $-1 < x \le 3$.

Ex. Determine the interval of convergence of

(i) $\sum (-1)^n \dfrac{x^{2n+1}}{(2n+1)}$ (ii) $\sum \dfrac{(-1)^{n+1}}{n}(x-1)^n$

(iii) $\sum \dfrac{n!\,(x+2)^n}{n^n}$

Sol. **(i)** a_n is coefficient of x^n. Thus $a_n = 1/(2n+1)$ is wrong. Actually $a_{2n+1} = 1/(2n+1)$ and $a_n = 0$ whenever n is even. Thus, we apply $R = 1 \div \limsup |a_n|^{\frac{1}{n}}$.

$$\limsup |a_n|^{\frac{1}{n}} = \lim \left|\frac{1}{2n+1}\right|^{\frac{1}{2n-1}} = 1.$$

Hence, R = 1.

The interval of convergence is (–1, 1).

For x = 1, the series becomes

$1 - \frac{1}{3} + \frac{1}{5} - \ldots$, which is convergent by Leibnitz's test.

For x = –1, the series becomes

$-1 + \frac{1}{3} - \frac{1}{5} + \ldots$, which is again convergent.

Hence the Exact interval of convergence is [–1, 1].

(ii) $\lim\limits_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim\limits_{n\to\infty}\left|\frac{n}{n-1}\right| = 1.$

$\therefore$ R = 1.

Since the given power series is about the point x = 1, the interval of convergence is $-1 + 1 < x < 1 + 1$ or $0 < x < 2$.

For x = 2, the given series is $\sum \frac{(-1)^{n+1}}{n}$, which is convergent by Leibnitz's Test.

Hence the exact interval of convergence is (0, 2].

(iii) The given power series is about the point x = –2.

$$\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty} \frac{n!}{n^n}\frac{(n+1)^{n+1}}{(n+1)!} = \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n$$

$$= \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e.$$

$\therefore$ R = e and the interval of convergence is (– 2 – e, –2 + e).

Theorems of Power Series

Theorem : If a power series $\Sigma a_n x^n$ converges for $x = x_0$, then

(i) it is absolutely convergent in the interval $|x| < |x_0|$

(ii) it is uniformly convergent in the interval $|x| \le |x_1|$, where $|x_1| < |x_0|$.

Proof. (i) Since $\Sigma a_n x_0^n$ converges, so $\lim\limits_{n\to\infty} a_n x_0^n = 0$.

Consequently there exists a positive integer m such that

$$|a_n x_0^n - 0| < 1 \ \forall\ n \ge m \text{ or } |a_n| \ \frac{1}{|x_0|^n} \ \forall\ n \ge m \qquad \ldots(1)$$

$$\Rightarrow \quad |a_n x^n| = |a_n|\,|x|^n < \frac{|x|^n}{|x_0|^n} \quad \forall\ n \geq m, \text{ by (1)}$$

$$\Rightarrow \quad |a_n x^n| < \left\{\frac{|x|}{|x_0|}\right\}^n \quad \forall\ n \geq n.$$

The series on the R.H.S. converges for $|x| < |x_0|$ (being a geometric series with common ratio < 1), it follows by the comparison test that $\Sigma\, |a_n x^n|$ is convergent for $|x| < |x_0|$.

Hence $\Sigma\, a_n x^n$ is absolutely convergent for $|x| < |x_0|$.

(ii) Let $M_n = \frac{|x_1|^n}{|x_0|^n}$. Then $\Sigma\, M_n$ converges, since $|x_1| < |x_0|$ (being a geometric series with common ratio < 1).

Using (1), we obtain

$$|a_n x^n| = |a_n|\ |x|^n < \frac{|x|^n}{|x_0|^n} < \frac{|x_1|^n}{|x_0|^n},$$

$\therefore \quad |a_n x^n| < M_n \ \forall\ n \geq m$, where $\Sigma\, M_n$ converges.

By the Weierstrass M-test, $\Sigma a_n x^n$ is uniformly convergent in $|x| < |x_1|$.

Note : If a power series whose radius of convergence is 1, then it converges absolutely $(-1, 1)$ and uniformly in $[-k, k]$, $|k| < 1$.

Theorem : If a power series $\Sigma a_n x^n$ converges for $|x| < R$, then it converges uniformly on $[-R + \varepsilon, R - \varepsilon]$, for each $\varepsilon > 0$.

Proof. Let $x_0 = R - \varepsilon < R$ so that $\Sigma a_n x_0^n$ converges absolutely.

For $\quad |x| < x_0$, $|a_n x^n| \leq |a_n|\, x_0^n$.

Since $\Sigma a_n x_0^n$ converges absolutely, therefore, by Weierstrass's M-test, the series $\Sigma a_n x^n$ converges uniformly on

$$[-x_0, x_0] = [-R + \varepsilon, R - \varepsilon].$$

Hence the theorem.

Theorem : Suppose a power series $\Sigma\, a_n x^n$ converges for $|x| < R$ and that $f(x) = \Sigma\, a_n x^n$. Then

(i) $f(x)$ is continuous in $(-R, R)$.

(ii) $\Sigma\, a_n x^n$ can be integrated term by term on $(-R, R)$ and the integrated series has the same radius of convergence as $\Sigma\, a_n x^n$.

(iii) $\Sigma\, a_n x^n$ can be differentiated term and term on $(-R, R)$ and $f(x) = \Sigma n\, a_n x^{n-1}$, $|x| < R$. Further the differential series has the same radius of convergence as $\Sigma\, a_n x^n$.

Proof. (i) Since each term of the power series $\Sigma a_n x^n$ is continuous in $(-R, R)$ and the series is uniformly convergent on $(-R, R)$, so the sum $f(x)$ of the given series is continuous in $(-R, R)$

(ii) Since each term of $\Sigma\, a_n x^n$ is continuous in $(-R, R)$ and $\Sigma\, a_n x^n$ converges uniformly to $f(x)$ in $(-R, R)$, so $\Sigma\, a_n x^n$ can be integrated term by term in $(-R, R)$ and the integrated series is $\sum \frac{a_n}{n+1} x^{n+1}$.

Further

$$\lim_{n\to\infty}\left|\frac{u_n}{u_{n+1}}\right| = \lim_{n\to\infty}\left(\frac{n+2}{n+1}\,\frac{a_n}{a_{n+1}}\right) = \lim_{n\to\infty}\frac{a_n}{a_{n+1}} = R,$$

which shows that $\Sigma a_n x^n$ and $\sum \frac{a_n x^{n+1}}{n+1}$ have the same radius of convergence.

(iii) We know each term of $\Sigma\, a_n x^n$ is continuous, has continuous derivative in $(-R, R)$ and $\Sigma\, a_n x^n$ is uniformly convergent in $(-R, R)$. Thus the series obtained by differentiating the power series $\Sigma a_n x^n$ is $\Sigma n\, a_n\, x^{n-1}$. Further

$$\lim_{n\to\infty}\left|\frac{u_n}{u_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{n\, a_n}{(n+1)a_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = R.$$

Hence the differentiated series $\Sigma n\, a_n x^{n-1}$ is also a power series which has the same radius of convergence as $\Sigma a_n x^n$.

Note. If $f(x) = \Sigma a_n x^n$, then $f(x)$ is called the **sum function of** the series.

Abel's Theorem : If a power series converges at an end point of its interval of convergence, then the power series is uniformly convergent in the interval which includes this end point.

Remarks : If a power series with interval of convergence $(-R, R)$ converges at both the end-points $-R$ and R, then the power series is uniformly convergent in $[-R, R]$.

Theorem : If a power series converges at the end-point $x = R$ of the interval of convergence $(-R, R)$, then it is uniformly convergent in $(-R, R]$ or $[-R + \varepsilon, R]$ where $\varepsilon > 0$. In particular, the series is uniformly convergent in $[0, R]$.

Abel's Limit Theorem : Let $\Sigma a_n x^n$ be a power series with finite radius of convergence R such that $f(x) = \Sigma a_n x^n - R < x < R$. If the series $\Sigma a_n R^n$ converges, then

$$\lim_{x\to R-} f(x) = \Sigma a_n R^n$$

Theorem : (Cauchy's product of two power series)

Two power series $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} b_n x^n$ can be multiple to obtain the power series $\sum_{n=0}^{\infty} c_n x^n$, where

$$c_n = a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \ldots\ldots + a_n b_n$$

the result being valid for each x within the common interval of convergence.

Ex. Show that the series $\Sigma x^n/n^2$ is uniformly convergent in $[-1, 1]$.

Sol. We have $a_n = \frac{1}{n^2}$, $a_{n+1} = \frac{1}{(n+1)^2}$ so that

$$\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\frac{(n+1)^2}{n^2} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^2 = 1$$

The radius of convergence is $R = 1$ and so the given series is uniformly convergent in $(-1, 1)$.

For $x = 1$, given series is $\sum \frac{1}{n^2}$, which is convergent.

For $x = -1$, given series becomes

$$-1+\frac{1}{2^2}-\frac{1}{3^2}+\frac{1}{4^2}-\ldots$$

This is an alternating series for which $u_{n+1} \le u_n\ \forall\ n$ and

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{n^2} = 0$$

So by Leibnitz's test, the series to convergent for x = –1. Hence by Abel's Theorem, given power series is uniformly convergent in [–1, 1].

Ex. Show that the series

$$1+\frac{x}{2}+\frac{x^2}{3}+\frac{x^3}{4}+\ldots$$

is uniformly convergent in [–1, k], 0 < k < 1.

Sol. We have

$$\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\left(\frac{n+1}{n}\right) = \lim_{n\to\infty}\left(1+\frac{1}{n}\right) = 1.$$

The radius of convergence is R = 1 and so the given series is absolutely convergent in]–1, 1[.

For x = 1, given series $\sum\frac{1}{n}$, which is divergent.

However for x = 1, given series becomes $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots$, which is convergent by Leibnitz's Test. Hence by Abel's Theorem, given power series is uniformly convergent [–1, 1[or [–1, k], where 0 < k < 1.

Ex. Show that the following series

$$x+\frac{1}{2}\frac{x^3}{3}+\frac{1.3}{2.4}\frac{x^5}{5}+\frac{1.3.5}{2.4.6}\frac{x^7}{7}+\ldots$$

is uniformly convergent in [–1, 1].

Sol. Ignoring the first term, we have

$$a_n = \frac{1.3.5.....(2n-1)}{2.4.6.....2n(2n+1)}, a_{n+1} = \frac{1.3.5......(2n-1)(2n+1)}{2.4.6......2n(2n+2)(2n+3)}$$

$$\therefore \quad \lim_{n\to\infty}\frac{a_n}{a_{n+1}} = \lim_{n\to\infty}\frac{(2n+2)(2n+3)}{(2n+1)^2} = 1.$$

Since the radius of convergence is R = 1, given power series is absolutely convergent in (–1, 1).
For x = 1, we see that the D'Alembert's ratio test fails. Now

$$\lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}}-1\right) = \lim_{n\to\infty}\frac{n(6n+5)}{(2n+1)^2} = \frac{6}{5} > 1.$$

By Raabe's test, the series is convergent for x = 1.

Again for x = –1, the series is convergent.

($\because$ Σa_n is convergent $\Rightarrow$ – Σa_n is convergent)

Hence the given power series is uniformly convergent in [–1, 1] (Abel's Theorem).

Ex. Prove that $\sum_{n=1}^{\infty}\frac{1.3.5......(2n-1)}{2.4.6......2n}x^n$ is uniformly convergent in [–1, 1).

Sol. Both the series given on the R.H.S. of (1) and (2) absolutely convergent in (–1, 1), so their

Cauchy's product converge to $(1-x)^{-1}\log(1-x)$. Thus

$$(1-x)^{-1}\log(1-x) = -\left[x+\left(1+\frac{1}{2}\right)x^2+\left(1+\frac{1}{2}+\frac{1}{3}\right)x^2+\ldots\right];\ -1<x<1$$

Integrating, we get

$$\frac{1}{2}[\log(1-x)]^2 = \frac{x^2}{2}+\left(1+\frac{1}{2}\right)x^3+\left(1+\frac{1}{2}+\frac{1}{3}\right)\frac{x^4}{4}+\ldots\quad -1<x<1$$

where the constant of integration vanishes.

Since the series on the right converges at $x = -1$, so by Abel's theorem, the above Expansion is valid for $-1 < x < 1$.

Ex. Show that

(i) $\tan^{-1} x = x - \frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\ldots,\ -1 \le x \le 1$

(ii) $\frac{\pi}{4} = 1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots.$

(iii) $\frac{1}{2}(\tan^{-1}x)^2 = \frac{x^2}{2}-\frac{x^4}{4}\left(1+\frac{1}{3}\right)+\frac{x^6}{6}\left(1+\frac{1}{3}+\frac{1}{5}\right)+\ldots,-1<x\le 1,$

Sol. We have

$$(1+x^2)^{-1} = 1-x^2+x^4-x^6+\ldots,\ |x|<1. \qquad \ldots(1)$$

Radius of convergence is

$$R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n-1}}\right| = 1.$$

Thus the series is convergent absolutely for all x such that $|x| < 1$, and is uniformly convergent in $[-k, k]$, $|k| < 1$.

The integrated series is also convergent absolutely in $(-1, 0)$ and uniformly in $[-k, k]$, $|k| < 1$.

Integrating (1), we get

$$\tan^{-1}x = \left(x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\ldots\right)+C,\ |x|<1$$

where C is constant of integration.

Putting $x = 0$, we get $C = 0$ and so

$$\tan^{-1}x = x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\ldots,\ |x|<1.$$

Now the series

$$x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\ldots \text{ is convergent at } x = \pm 1.$$

[$\because$ for $x = 1$, $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}\ldots$ is an alternating series which converges, by Leibnitz's test. Similarly it is convergent for $x = -1$].

By Abel's theorem, it is uniformly convergent in [–1, 1].

Hence $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + ...,\ -1 \le x \le 1.$

(ii) Putting x = 1, by Abel's limit theorem,

$$\frac{\pi}{4} = \tan^{-1} 1 = \lim_{x \to 1-0} \tan^{-1} x = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + ...$$

(iii) We have

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + ...,\ -1 \le x \le 1$$

and $(1 + x^2)^{-1} = 1 - x^2 + x^4 - x^6 + ...,\ -1 < x < 1.$

By part (i), both the series are absolutely convergent in (–1, 1), so they can be multiplied and their Cauchy's product will be equal to

$$(1+x^2)^{-1} \tan^{-1} x = x - \left(1+\frac{1}{3}\right)x^2 + \left(1+\frac{1}{3}+\frac{1}{5}\right)x^5 - ..... -1 < x < 1.$$

integrating, we get

$$\frac{1}{2}(\tan^{-1} x)^2 = \frac{x^2}{2} - \frac{x^4}{4}\left(1+\frac{1}{3}\right) + \frac{x^6}{6}\left(1+\frac{1}{3}+\frac{1}{5}\right) - .... -1 < x < 1$$

Here is the constant of integration vanishes on taking x = 0.

By Leibnitz's test, the series on the R.H.S. converges for x = 1. Hence the above expansion is valid for –1 < x < 1.

Ex. Show that

$$\sin^{-1} x = x + \frac{1}{2}\frac{x^3}{3} + \frac{1.3}{2.4}\frac{x^5}{5} + \frac{1.3.5}{2.4.6}\frac{x^7}{7} + ...,\ -1 < x < 1.$$

Deduce that

$$\frac{\pi}{2} = 1 + \frac{1}{2}\cdot\frac{1}{3} + \frac{1.3}{2.4}\cdot\frac{1}{5} + \frac{1.3.5}{2.4.6}\cdot\frac{1}{7} + ...$$

Sol. We know that

$$\frac{1}{\sqrt{1-x^2}} = (1-x^2)^{-1/2}$$

$$= 1 + \frac{1}{2}x^2 + \frac{1.3}{2.4}x^4 + \frac{1.3.5}{2.4.6}x^5 + ...,\ |x| < 1$$

The series on the R.H.S. is a power series with radius of convergence 1 and so it is absolutely convergent in(–1, 1) and uniformly convergent for [–k, k], | k | < 1. The integrated series is also convergent in (–1, 1) and uniformly convergent for [–k , k], | k | < 1. integrating, we get

$$\sin^{-1} x = c + x + \frac{1}{2}\cdot\frac{x^3}{3} + \frac{1.3}{2.4}\cdot\frac{x^5}{5} + \frac{1.3.5}{2.4.6}\cdot\frac{x^7}{7} + ...,|x| < 1$$

Putting x = 0, we get c = 0.

$\therefore \quad \sin^{-1} x = x + \frac{1}{2} \cdot \frac{x^3}{3} + \frac{1.3}{2.4} \cdot \frac{x^5}{5} + \frac{1.3.5}{2.4.6} \cdot \frac{x^7}{7} + \ldots, -1 < x < 1$ ...(1)

For x = 1, the nth term of the series on the R.H.S. of (1) is

$$u_n = \frac{1.3.5....(2n-1)}{2.4.6.....2n\ (2n+1)},$$

$$u_{n+1} = \frac{1.3.5....(2n+1)}{2.4.6.....(2n+2)(2n+3)}.$$

$$\therefore \quad \frac{u_n}{u_{n+1}} = \frac{(2n+2)(2n+3)}{(2n+1)^2},\ n\left(\frac{u_n}{u_{n+1}} - 1\right) = \frac{6n^2+5}{(2n+1)^2}$$

$$\lim_{n \to \infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = \frac{3}{2} > 1.$$

So by Raabe's test, the series is convergent for x = 1. Hence

$$\sin^{-1} x = x + \frac{1}{2} \cdot \frac{x^3}{3} + \frac{1.3}{2.4} \frac{x^5}{5} + \frac{1.3.5}{2.4.6} \frac{x^7}{7} + \ldots,$$

(ii) At x = 1, by Abel's limit theorem

$$\frac{\pi}{2} = \lim_{x \to 1} \sin^{-1} x = 1 + \frac{1}{2} \cdot \frac{1}{3} + \frac{1.3}{2.4} \cdot \frac{1}{5} + \frac{1.3.5}{2.4.6} \cdot \frac{1}{7} + \ldots$$

Ex. Prove that

$$\int_0^x \frac{dt}{1+t^n} = x - \frac{x^{n+1}}{n+1} + \frac{x^{(2n+1)}}{2n+1} - \ldots,\ -1 < x \le 1,\ x > 0.$$

Sol. For n > 0, we have

$(1 + t^n)^{-1} = 1 - t^n + t^{2n} - t^{3n} + \ldots$

It is clear that the series on the right has its radius of convergence R = 1.

By Theorem this series can be integrated term by term on (–1, 1) and the integrated series will also have the radius of convergence as unity. Thus

$$\int_0^x \frac{dt}{1+t^n} = x - \frac{x^{n+1}}{n+1} + \frac{x^{2n+1}}{2n+1} - \ldots, -1 < x < 1. \quad \ldots(1)$$

At x = 1, the series on the right is convergent, by Leibnitz's test.

Hence (1) is true for $-1 < x \le 1$.

Theorem (Differentiation of Power Series)

Let a power series

$$\sum_{n=1}^{\infty} a_n (x-c)^n$$

have non-zero radius of convergence R and

$$f(x) = \sum_{n=1}^{\infty} a_n (x-c)^n,\ x \in (c - R, c + R)$$

Then, the following holds :

The function f is differentiable on the interval (c – R, c + R). Further the series $\sum_{n=1}^{\infty} \frac{d}{dx}(a_n(x-c)^n)$ also has radius of convergence R and $f(x) = \sum_{n=1}^{\infty} \frac{d}{dx}(a_n(x-c)^n)$, $x \in (c - R, c + R)$.

The function f has derivatives of all orders and $f^{(k)}(x) = \sum_{n=1}^{\infty} \frac{d^k}{dx^k}(a_n(x-c)^n)$, $x \in (c - R, c + R)$.

Theorem (Integration of Power Series)

Let

$$\sum_{n=1}^{\infty} a_n(x-c)^n$$

be a power series with non-zero radius of convergence R and let

$$f(x) = \sum_{n=1}^{\infty} a_n(x-c)^n, \ x \in (c - R, c + R)$$

Then

(i) The function f has an anti derivative F(x) given by

$$F(x) = \int f(x)dx = \sum_{n=1}^{\infty} \frac{a_n(x-c)^{n+1}}{n+1} + C,$$

where C is an arbitrary constant and the series on the right hand side has radius of convergence R.

(ii) For $[\alpha, \beta] \subset (c - R, c + R)$,

$$\int_{\alpha}^{\beta} f(x)dx = \sum_{n=1}^{\infty}\left[\int_{\alpha}^{\beta} a_n(x-c)^n dx\right],$$

where the series on the right hand side is absolutely convergent.

Example : Consider the power series

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}$$

By the ratio test, for every x

$$\lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)!}\frac{n!}{x^n}\right| = \lim_{n\to\infty}\frac{|x|}{n+1} = 0$$

Hence, the series is absolutely convergent for every x. Let

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \ x \in \mathbf{R}$$

Then, f is differentiable by theorem and

$$f'(x) = \sum_{n=0}^{\infty} \frac{d}{dx}\left(\frac{x^n}{n!}\right) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = f(x).$$

Some Important Power Series

Consider the following power series :

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \ldots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + (-1)^n \frac{x^{2n}}{(2n)!} + \ldots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \ldots$$

For the first series, $\lim\limits_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim\limits_{n\to\infty} \frac{(n+1)!}{n!} = \lim\limits_{n\to\infty} (n+1) = \infty.$

Theorem : Define cosine and sine functions as sums of power series. Prove that

(i) $S(x + y) = S(x)C(y) + C(x)S(y)$,

(ii) $C(x + y) = C(x)C(y) - S(x)S(y)$,

where C, S denote cosine and sine respectively.

Proof. The cosine function is defined as

$$C(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + (-1)^n \frac{x^{2n}}{(2n)!} + \ldots \ \forall\ x \in \mathbf{R} \qquad \ldots(1)$$

The sine function is defined as

$$S(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^n \frac{x^{2n-1}}{(2n+1)!} + \ldots \ \forall\ x \in \mathbf{R} \qquad \ldots(2)$$

These two power series are uniformly convergent $\forall\ x \in \mathbf{R}$ and consequently, the functions C(x) and S(x) are continuous $\forall\ x \in \mathbf{R}$

Differentiating term by term the convergent series (2),

$$S'(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots$$

which gives us the uniformly convergent series (1),

Thus S(x) is derivable, $\forall\ x \in \mathbf{R}$ and $S'(x) = C(x)$. ...(3)

Similarly $C'(x) = -S(x)$. ...(4)

From (1) and (2); $S(0) = 0$ and $C(0) = 1$. ...(5)

Let y be any arbitrary but fixed real number. We write

$$f(x) = S(x + y) - S(x)C(y) - C(x)S(y), \qquad \ldots(6)$$

$$g(x) = C(x + y) - C(x)C(y) + S(x)S(y). \qquad \ldots(7)$$

Differentiating w.r.t. x, we obtain

$$f'(x) = C(x + y) - C(x)C(y) + S(x)S(y) = g(x),$$

$$g'(x) = -S(x + y) + S(x)C(y) + C(x)S(y) = -f(x). \qquad \text{[Using (3), (4)]}$$

Now $\frac{d}{dx}[f^2(x) + g^2(x)] = 2f(x)f'(x) + 2g(x)g'(x)$

$= 2f(x)g(x) - 2g(x)f(x) = 0\ \forall\ x \in \mathbf{R}$

It follows that $f^2(x) + g^2(x)$ is a constant $\forall\ x \in \mathbf{R}$

$\Rightarrow \quad f^2(x) + g^2(x) = f^2(0) + g^2(0) \ \forall \ x \in \mathbf{R}$

$$= [S(y) - S(0)C(y) - C(0)S(y)]^2 + [C(y) - C(0)C(y) + S(0)S(y)]^2$$

$$= 0 \qquad \text{[Using (5)]},$$

$\therefore \ f^2(x) + g^2(x) = 0, \ \forall \ x \in \mathbf{R}$ implies that $f(x) = 0$ and $g(x) = 0$.

Hence (6) and (7) yield

$$S(x + y) = S(x)C(y) + C(x)S(y),$$

$$C(x + y) = C(x)C(y) - C(x)S(y).$$

Corollary 1. $S(-x) = -S(x)$; $C(-x) = C(x) \ \forall \ x \in \mathbf{R}$

We have $\quad S(-x) = -x - \dfrac{(-x)^3}{3!} + \dfrac{(-x)^5}{5!} - \ldots$

$$= -\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\right] = -S(x).$$

Similarly we can prove the second part.

Corollary 2. $S(x - y) = S(x)C(y) - C(x)S(y)$,

$$C(x - y) = C(x)C(y) + S(x)S(y).$$

Replacing y by $-$ y in (i) and (ii) of Theorem

$$S(x - y) = S(x)C(-y) + C(x)S(-y),$$

$$C(x - y) = C(x)C(-y) - S(x)S(-y).$$

The desired results follow by Corollary 1.

Corollary 3. $C^2(x) + S^2(x) = 1$ or $\cos^2 x + \sin^2 x = 1 \ \forall \ x \in \mathbf{R}$

We have $\quad 1 = C(0) = C(x - x)$

$$= C(x)C(-x) - S(x)S(-x)$$

$$= C^2(x) + S^2(x), \text{ by Corollary 1}$$

Hence $\cos^2 x + \sin^2 x = 1 \ \forall \ x \in \mathbf{R}$

Corollary 4. $| C(x) | \le 1$, $| S(x) | \le 1$. $\forall \ x \in \mathbf{R}$

$$| \cos x | \le 1, \ | \sin^2 x | \le 1 \ ; \ \forall \ x \in \mathbf{R}$$

This follows by Corollary 3.

Corollary 5. $\sin 2x = 2 \sin x \cos x$, $\cos 2x = \cos^2 x - \sin^2 x$.

Replacing y by x in (i) and (ii) of Theorem we obtain

$$S(2x) = 2S(x)\ C(x), \ C(2x) = C^2(x) - S^2(x)$$

Hence the results follows.

Remark. It may be noted that the above properties of C(x) and S(x) are similar to those of the trigonometric functions cos x and sin x respectively. Thus we have

$$C(x) \equiv \cos x \quad \text{and} \quad S(x) \equiv \sin x.$$

Theorem : Define cos x as a power series and determine its domain. Prove that there exists a least positive number π such that $\cos\left(\dfrac{1}{2}\pi\right) = 0$

Proof. cos x is defined as the sum of the power series

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + (-1)^n \frac{x^{2n}}{(2n)!} + \ldots$$

Its radius of convergence is

$$R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{(2n+2)!}{(2n+1)!}\right| = \lim_{n\to\infty}(2n+2) = \infty.$$

Thus this power series is convergent for $\forall\ x \in \mathbf{R}$ and so the domain of cos x is **R**.

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} + \ldots\ \forall\ x \in \mathbf{R} \qquad \ldots(1)$$

$\therefore$ $\cos 0 = 1 > 0$ and

$$\cos 2 = 1 - \frac{2^2}{2!} + \frac{2^4}{4!} - \frac{2^6}{6!} + \frac{2^8}{8!} + \ldots$$

$$= 1 - \frac{2^2}{2!}\left(1 - \frac{2^2}{3.4}\right) - \frac{2^6}{6!}\left(1 - \frac{2^8}{7.8}\right) - \ldots$$

Since the brackets are all positive, therefore

$$\cos 2 < 1 - \frac{2^2}{2!}\left(1 - \frac{2^2}{3.4}\right) = -\frac{1}{3} \text{ and so } \cos 2 < 0.$$

Thus we see that cos x is a continuous function in the closed interval [0, 2], where cos 0 and cos 2 are of opposite signs. By Intermediate value theorem, there exist at least one number $\alpha \in\ ]0, 2[$ such that $\cos \alpha = 0$. We now proceed to show that α is unique.

Let, if possible, $\beta \in (0, 2)$ satisfy $\cos \beta = 0$ so that

$\cos \alpha = \cos \beta = 0,\ 0 < \alpha, \beta < 2.$

By Rolle's theorem, there exists some number λ, $\alpha < \lambda < \beta$, such that $-\sin \lambda = 0$.

$\therefore$ $\sin \lambda = 0,\ 0 < \lambda < 2.$ ...(2)

By definition,

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots..\ \forall\ x \in \mathbf{R}$$

$$\therefore \quad \sin \lambda = \frac{\lambda}{1!}\left(1 - \frac{\lambda^2}{2.3}\right) + \frac{\lambda^5}{5!}\left(1 - \frac{\lambda^2}{6.7}\right) + \ldots > 0, \text{ for } 0 < \lambda < 2.$$

This contradicts (2).

Thus there exists one and only one root of the equation cos x = 0 lying between 0 and 2. i.e., $\cos \alpha = 0,\ 0 < \alpha < 2$.

We denote α as $\frac{1}{2}\pi$. Hence $\frac{1}{2}\pi$ is the least positive root of the equation cos x = 0 i.e., $\cos\left(\frac{1}{2}\pi\right) = 0$.

Also, therefore, we have $\cos x > 0$ when $0 \le x < \frac{1}{2}\pi$.

Theorem : Define Exponential function E(x) as the sum of a power series. Show that the domain is the set of all real numbers. Prove that

$E(x + y) = E(x)E(y)$ for all $x, y \in \mathbf{R}$.

If e denote E(1), prove that $E(x) = E^x$ for all real x.

Proof. The Exponential function E(x) is defined as the sum of the power series

$$E(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + \dots \qquad \dots(1)$$

We have $\quad R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty} \frac{(n+1)!}{n!} = \lim_{n\to\infty} (n+1) = \infty.$

Thus the radius of convergence of the above power series is $R = \infty$ so that the power series is everywhere convergent i.e., E(x) is defined $\forall\ x \in \mathbf{R}$. This shows that the domain of the Exponential function is **R**.

It is clear that the function E(x), defined by (1), is continuous and has derivatives of all orders, for each $x \in \mathbf{R}$.

Differentiating (1), we get

$E'(x) = E(x)$ and so $E''(x) = E(x)$, $E'''(x) = E(x)$ etc.

in fact $E^{(n)}(x) = E(x) \quad \forall\ n \in \mathbf{N}$.

By Taylor's Theorem, for all real values t and x

$$E(t) = E(x) + \frac{E'(x)}{1!}(t-x) + \frac{E''(x)}{2!}(t-x)^2 + \dots$$

or $$E(t) = E(x) + \frac{E(x)}{1!}(t-x) + \frac{E(x)}{2!}(t-x)^2 + \dots$$

Replacing t by x + y, we get

$$E(x+y) = E(x)\left\{1 + \frac{y}{1!} + \frac{y^2}{2!} + \dots\right\} = E(x)E(y).$$

Thus $E(x + y) = E(x)\, E(y)\ \forall\ x, y \in \mathbf{R}$. $\qquad \dots(2)$

From (1), $\quad E(1) = 1 + \frac{1}{1!} + \frac{1}{2!} + \dots + \frac{1}{n!} + \dots$

The above series on the right converges to a number which lies between 2 and 3.

We denote this number by e so that E(1) = e.

The property (2) can be Extended as

$$E(x_1 + x_2 + \dots + x_n) = E(x_1)\, E(x_2) \dots E(x_n) \qquad \dots(3)$$

for any positive integer n and $\forall\ x \in \mathbf{R}$.

Taking $x_1 = x_2 = \dots = x_n = x$ in (3), we obtain

$$E(nx) = \{E(x)\}^n \quad \forall\ x \in R,\ \forall\ n \in \mathbf{N}. \qquad \dots(4)$$

For x = 1, we obtain

$$E(n) = e^n \quad \forall\ n \in \mathbf{N} \qquad (\because E(1) = e) \qquad \dots(5)$$

Taking x = m/n (m and n being positive integers) in (4),

$E(m) = \{E(m/n)\}^n$ or $e^m = \{E(m/n)\}^n$, by (5)

$\therefore \quad E(m/n) = (e^m)^{1/n} = e^{m/n}$.

Thus $E(q) = e^q$ for all positive rationals q. $\qquad \dots(6)$

Let x be any positive irrational number. Then there always exists a sequence $\{x_n\}$ of positive rational numbers such that $x_n \to x$.

Using (6), $E(x_n) = e^{x_n}\ \forall\ n \in \mathbf{N}$.

Since $x_n \to x$ as $n \to \infty$ and E is a continuous function $\forall\ x \in \mathbf{R}$, therefore

$$E(x) = e^x \text{ for all positive irrationals.} \quad ...(7)$$

From (6) and (7), we conclude that

$$E(x) = E^x \text{ for all positive reals.} \quad ...(8)$$

Taking $y = -x$ in (2), we get

$$E(0) = E(x)E(-x) \text{ or } 1 = E(x)\,E(-x) \quad (x \geq 0) \qquad [\because E(0) = 1, \text{ by } (1)]$$

or $\quad E(-x) = \dfrac{1}{E(x)} = \dfrac{1}{e^x} = e^{-x}$, using (8)

Hence $E(x) = E^x\ \forall\ x \in \mathbf{R}$.

Theorem : Let f be a continuous function on R such that

$$f(x + y) = f(x) + f(y)\ \forall\ x, y \in \mathbf{R}. \quad ...(1)$$

Show that $f(x) = cx$ for some constant c and $\forall\ x \in \mathbf{R}$.

Proof. Putting $y = x$ in (1), $f(2x) = 2f(x)$.

In fact $f(nx) = nf(x)\ \forall\ x \in \mathbf{R},\ \forall\ n \in \mathbf{N}$.

Replacing x by x/n, we get

$$f(x) = nf(x/n) \quad \text{or} \quad f(x/n) = (1/n)\,f(x)\ \forall\ x \in \mathbf{R}.$$

For any integers p and p(q > 0), it follows that

$$f\left(\frac{p}{q}x\right) = \frac{p}{q}f(x).\ \forall\ x \in \mathbf{R}.$$

Thus $f(rx) = r\,f(x)$, for any rational number r. ...(2)

Let us write $f(1) = e$. Taking $x = 1$ in (2), we get

$$f(x) = cr, \text{ for every rational number } r. \quad ...(3)$$

Let a be any irrational number. Then there exists a sequence $\{r_n\}$ of rational numbers such that $r_n \to a$

$\Rightarrow \quad f(r_n) \to f(a)$, by continuity of f

$\Rightarrow \quad f(a) = \lim f(r_n) = \lim cr_n = c \lim r_n = ca$, using (3).

Hence $f(x) = cx\ \forall\ x \in \mathbf{R}$.

Theorem : Let f be a non-zero continuous function on **R** such that

$$f(x + y) = f(x)\,f(y)\ \forall\ x, y \in \mathbf{R}. \quad ...(1)$$

Show that $f(x) = e^{cx}$, for some constant $c \neq 0$.

Proof. It is given that $f(x) \neq 0\ \forall\ x \in \mathbf{R}$. From (1),

$$f(x) = f\left(\frac{1}{2}x\right)f\left(\frac{1}{2}x\right) \quad \text{or} \quad f(x) = \left\{f\left(\frac{1}{2}x\right)\right\}^2 > 0\ \forall\ x \in \mathbf{R}.$$

Let $g(x) = \log_e f(x)\ \forall\ x \in \mathbf{R}$. Then by (1),

$$g(x + y) = g(x) + g(y)\ \forall\ x, y \in \mathbf{R}. \quad ...(2)$$

Now g, being the composition of two continuous functions f(x) and $\log_e x$, is itself continuous on R and g satisfies (2).

By Theorem, it follows that

$$g(x) = cx\ \forall\ x \in \mathbf{R}, \text{ c being a constant}$$

$\Rightarrow \qquad \log_e f(x) = cx \quad \Rightarrow \quad f(x) = e^{cx}\ \forall\ x \in \mathbf{R}.$

Theorem : Let f be a non-zero function continuous on $\mathbf{R}^+$ such that $f(xy) = f(x) + f(y)\ \forall\ x, y > 0.$ ...(1)

Show that $f(x) = \log_a x,\ x \in \mathbf{R}^+\ (a > 0,\ a \neq 1).$

Proof. For $x > 0$, let us take $t = \log_e x \Leftrightarrow x = e^t,\ t \in \mathbf{R}$ ...(2)

Let $g(t) = f(e^t)\ \forall\ t \in \mathbf{R}$. Then g is continuous on **R** and

$$g(t + s) = f(e^{t+s}) = f(e^t e^s) + f(e^s) \qquad \text{by (1)}$$

or $$g(t + s) = g(t) + g(s).$$

Here $t, s \in \mathbf{R}$. By Theorem $g(t) = ct,\ \forall\ t \in \mathbf{R}$, c being a constant.

Thus $f(x) = c \log_e x\ \forall\ x \in \mathbf{R}^+$, where $c \neq 0$ as f is non-zero.

Let $a = e^{1/s}$. Then $f(x) = \log_a x\ \forall\ x \in \mathbf{R}^+\ (a > 0,\ a \neq 1).$

Taylor Series and Maclaurin Series

Definition : Let $f : I = (a - \delta, a + \delta) \to R$ be a function which has derivation of all order in I. Then the power series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{(n)!}(x-a)^n,$$

is called the Taylor series for f at x = a. We say f has Taylor series Expansion at x = a, if its Taylor series is convergent for $x \in I$ and its sum is f(x). For a = 0, the Taylor series for f is called the Maclaurin Series for f at x = 0.

Example : For the function $f(x) = \frac{1}{x}$, $x \neq 0$, its derivatives of all order exist in domain

$$I = (-\infty, 0) \cup (0, \infty)$$

For $a = 1$, since

$$f^{(n)}(x) = (-1)^n n! x^{-(n+1)}$$

we have

$$f^{(n)}(1) = (-1)^n n!,\ \forall\ n \geq 1$$

Thus $\sum_{n=0}^{\infty} (-1)^n (x-1)^n$

is its Taylor series at x = 1. Since it is a geometric series, it will be convergent if

$$|x - 1| < 1 \quad \text{i.e.,} \quad 0 < x < 2.$$

Further, its sum is

$$\frac{1}{1+(x-1)} = \frac{1}{x}$$

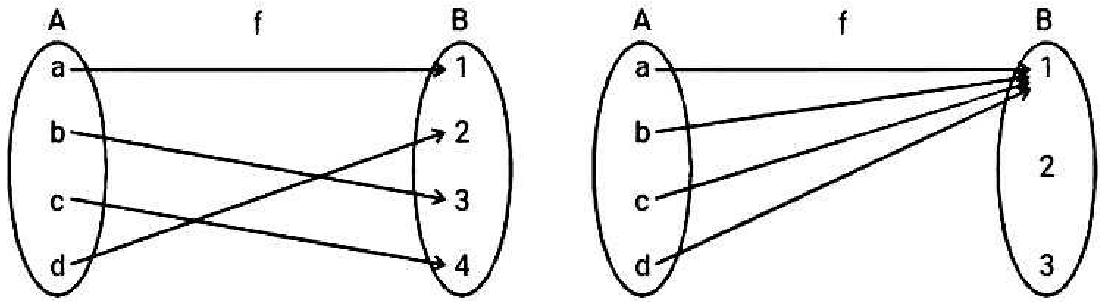
Hence, $f(x) = \frac{1}{x}$ has Taylor series Expansion

$$\frac{1}{x} = \sum_{n=0}^{\infty} (-1)^n (x-1)^n,\ 0 < x < 2.$$

Unit-4 Functions of One Variable-I

1. INTRODUCTION

Function: If A and B two non-empty sets then a relation defined from A to B is said to be a function if every element of A is associated with some unique element of set B



(i) (ii)

Domain, Co-domain & Range

If y = f(x) is a function such that f is defined from $A \to B$ then

(1) Domain

Set A is called domain of f(x) and it is the set from which the independent variable 'x' takes its values. The independent variables 'x' must be able to take each and every element of set A

(2) Co-domain

Set B is called co-domain of f(x) and it is the set from which the dependent variable y takes its values the dependent variables 'y' cannot take its values outside the co-domain.

(3) Range

The set of values that 'y' actually takes for different values of 'x' is called range of f(x)

$\Rightarrow$ Range is a subset of B

Range $\subseteq$ co-domain

Real-valued function: A function whose range is in the real numbers is said to be a real-valued function, also called a real function.

Ex : $f(x) = \dfrac{1}{x}$ a real-valued function.

Sol. Each member of the codomain of $f(x) = \dfrac{1}{x}$ is a real number.

Also note that excluding zero from the codomain does not change the fact that every other member is a real number. Symbolically, we have

$$f : \mathbb{R} \setminus \{0\} \to \mathbb{R} \setminus \{0\}$$

Since $\mathbb{R}\setminus\{0\}$ is a subset of $\mathbb{R}$, then the expression above can be read as f is a real-valued function of a real-valued variable.

Ex: The function f, g, and h defined on $(-\infty, \infty)$ by

$$f(x) = x^2, \quad g(x) = \sin x, \quad \text{and} \quad h(x) = e^x$$

have ranges $[0, \infty)$, $[-1, 1]$, and $(0, \infty)$, respectively.

Ex: The equation

$$[f(x)]^2 = x \quad \text{...(1)}$$

does not define a function except on the singleton set {0}. If $x < 0$, no real number satisfies (1), while if $x > 0$, two real numbers satisfy (1). However, the conditions

$$[f(x)]^2 = x \quad \text{and} \quad f(x) \geq 0$$

define a function f on $D_f = [0, \infty)$ with values $f(x) = \sqrt{x}$. Similarly, the conditions

$$[g(x)]^2 = x \quad \text{and} \quad g(x) \leq 0$$

define a function g on $D_g = (-\infty, 0]$ with values $g(x) = -\sqrt{x}$. The ranges of f and g are $[0, \infty)$ and $(-\infty, 0]$, respectively.

Arithmetic Operations on Functions

Definition : If $D_f \cap D_g \neq 0$, then $f + g$, $f - g$ and fg are defined by

$$(f + g)(x) = f(x) + g(x),$$

$$(f - g)(x) = f(x) - g(x),$$

and

$$(fg)(x) = f(x)g(x).$$

The quotient f/g is defined by

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$$

for x in $D_f \cap D_g$ such that $g(x) \neq 0$.

Ex: If $f(x) = \sqrt{4 - x^2}$ and $g(x) = \sqrt{x - 1}$, then $D_f = [-2, 2]$ and $D_g = [1, \infty)$, so $f + g$, $f - g$, and fg are defined on $D_f \cap D_g = [1, 2]$ by

$$(f + g)(x) = \sqrt{4 - x^2} + \sqrt{x - 1},$$

$$(f - g)(x) = \sqrt{4 - x^2} - \sqrt{x - 1},$$

and

$$(fg)(x) = (\sqrt{4 - x^2})(\sqrt{x - 1}) = \sqrt{(4 - x^2)(x - 1)}. \quad \text{...(2)}$$

The quotient f/g is defined on (1, 2] by

$$\left(\frac{f}{g}\right)(x) = \sqrt{\frac{4 - x^2}{x - 1}}.$$

Although the last expression in (2) is also defined for $-\infty < x < -2$, it does not represent fg for such x, since g is not defined on $(-\infty, -2]$ but f is.

Ex : If c is a real number, the function cf defined by $(cf)(x) = cf(x)$ can be regarded as the product of f and a constant function. Its domain is D_f. The sum and product of n (≥ 2) functions $f_1, \ldots, f_n$ are defined by

$$(f_1 + f_2 + \ldots + f_n)(x) = f_1(x) + f_2(x) + \ldots + f_n(x)$$

and

$$(f_1 \cdot f_2 \cdot \ldots \cdot f_n)(x) = f_1(x) \cdot f_2(x) \cdot \ldots \cdot f_n(x) \quad \text{...(3)}$$

on $D = \bigcap_{i=1}^{n} D_{f_i}$, provided that D is nonempty. If $f_1 = f_2 = \dots = f_n$, then (3) defines the nth power of f :

$$(f^n)(x) = (f(x))^n.$$

From these definitions, we can build the set of all polynomials

$$p(x) = a_0 + a_1x + \dots + a_nx^n,$$

starting from the constant functions and f(x) = x. The quotient of two polynomials is a rational function

$$r(x) = \frac{a_0 + a_1x + \dots + a_nx^n}{b_0 + b_1x + \dots + b_mx^m} \qquad (b_m \neq 0).$$

The domain of r is the set of points where the denominator is nonzero.

Domain of definition:

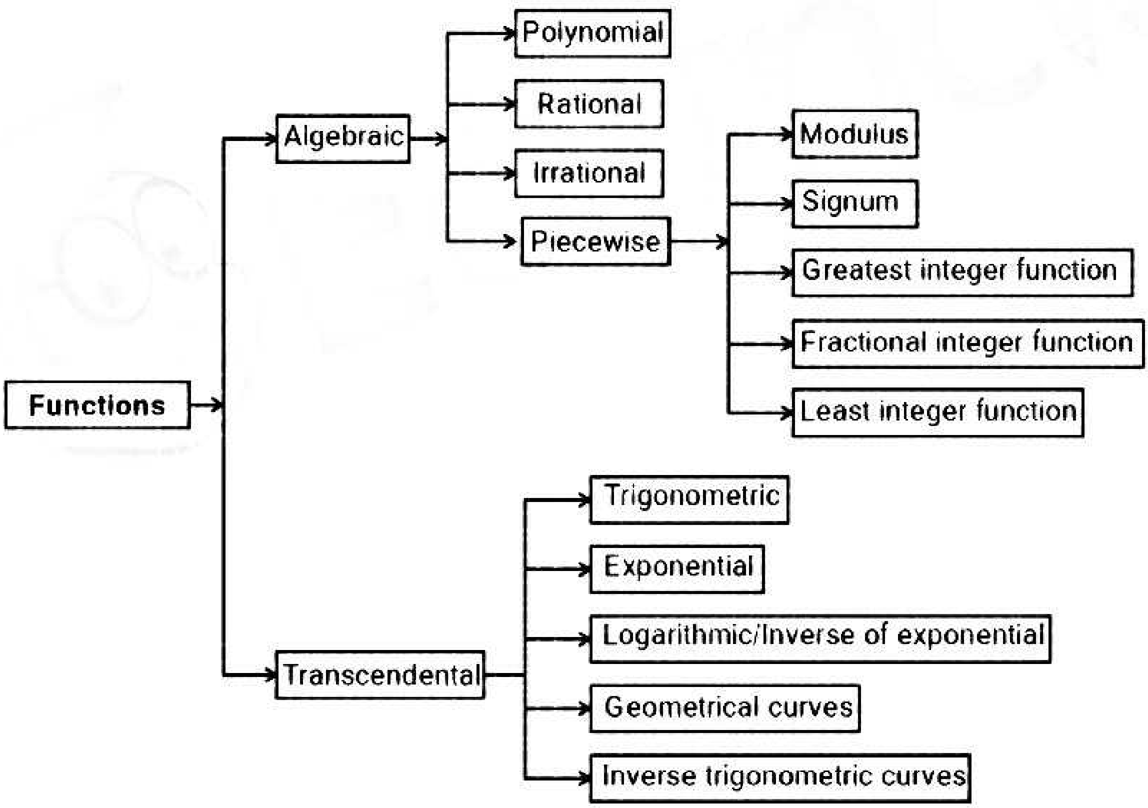
The domain on which the function is define is known as domain of definition.

[let y = f(x) be a rule, D subset of R on which f becomes real valued function, i.e. if $f : D \to R$ with D subset of R then D is called *Dod* - domain of definition

Ex- f(x) = (1/x), Dod ~ R-{0}

Ex- $f(x) = \log_e x$, Dod ~ R^+]

Types of functions



1. Algebraic Functions

1.1 Polynomial Function

A function of the form :

$$f(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n ;$$

where $n \in \mathbf{N}$ and $a_0, a_1, a_2, \dots, a_n \in \mathbb{R}$.

Then, f is called a polynomial function. "f(x) is also called polynomial in x".

Some of basic polynomial functions are

(i) Identity function/Graph of $f(x) = x$

A function f defined by $f(x) = x$ for all $x \in \mathbb{R}$, is called the identity function.

Here, $y = x$ clearly represents a straight line passing through the origin and inclined at an angle of 45° with x-axis shown as :

The domain and range of identity functions are both equal to $\mathbb{R}$.

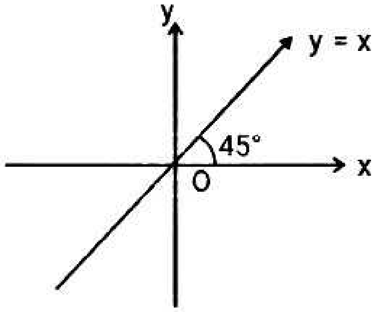


Fig.

(ii) Graph of $f(x) = x^2$

A function given by $f(x) = x^2$ is called the square function.

The domain of square function is $\mathbb{R}$ and its range is $\mathbb{R}^+ \cup \{0\}$ or $[0, \infty)$

Clearly $y = x^2$, is a parabola. Since $y = x^2$ is an even function, so its graph is **symmetrical about y-axis**, shown as :

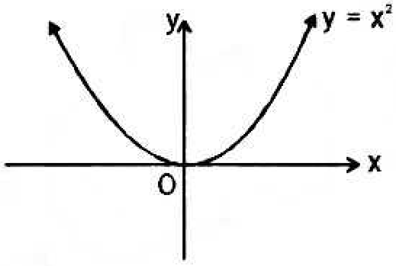


Fig.

(iii) Graph of $f(x) = x^3$

A function given by $f(x) = x^3$ is called the cube function.

The domain and range of cube are both equal to $\mathbb{R}$.

Since, $y = x^3$ is an odd function, so its graph is **symmetrical about opposite quadrant**, i.e., **"origin"**, shown as :

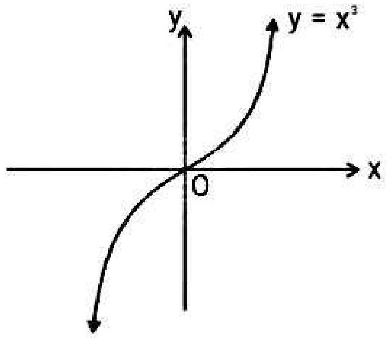


Fig.

(iv) Graph of $f(x) = x^{2n}$; $n \in N$

If $n \in \mathbf{N}$, then function f given by $f(x) = x^{2n}$ is an even function.

So, its graph is always **symmetrical about y-axis.**

Also, $x^2 > x^4 > x^6 > x^8 > \ldots$ for all $x \in (-1, 1)$

and $x^2 < x^4 < x^6 < x^8 < \ldots$ for all $x \in (-\infty, -1) \cup (1, \infty)$

Graphs of $y = x^2$, $y = x^4$, $y = x^6$, ..., etc. are shown as :

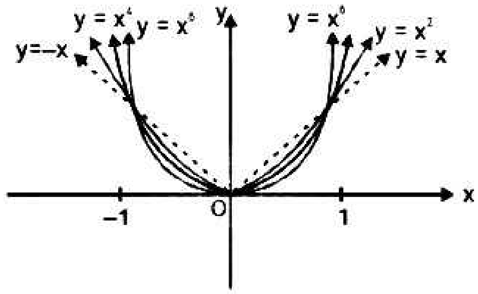


Fig.

(v) Graph of $f(x) = x^{2n-1}$; $n \in N$

If $n \in N$, then the function f given by $f(x) = x^{2n-1}$ is an odd function. So, its graph is **symmetrical about origin or opposite quadrants.**

Here, comparison of values of $x, x^3, x^5, \ldots$

$x \in (1, \infty)$	$x < x^3 < x^5 < \ldots$
$x \in (0, 1)$	$x > x^3 > x^5 > \ldots$
$x \in (-1, 0)$	$x < x^3 < x^5 < \ldots$
$x \in (-\infty, -1)$	$x > x^3 > x^5 > \ldots$

Graphs of $f(x) = x$, $f(x) = x^3$, $f(x) = x^5$, ... are shown as in Fig.

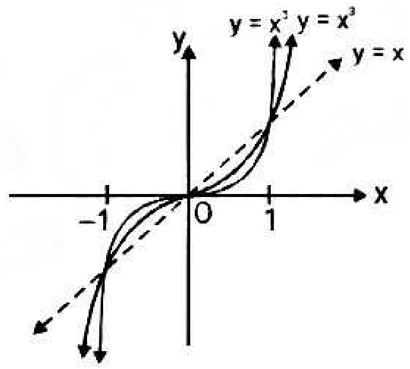


Fig.

1.2 Rational Expression

A function obtained by dividing a polynomial by an other polynomial is called a rational function.

$\Rightarrow \quad f(x) = \dfrac{P(x)}{Q(x)}$

Domain $\in \mathbb{R} - \{x \mid Q(x) = 0\}$

i.e. domain $\in \mathbb{R}$ except those points for which denominator = 0.

Graphs of some Simple Rational Functions

(i) Graph of $f(x) = \dfrac{1}{x}$

A function defined by $f(x) = \dfrac{1}{x}$ is called the reciprocal function or rectangular hyperbola, with coordinate axis as asymptotes. The domain and range of $f(x) = \dfrac{1}{x}$ is $\mathbb{R} - \{0\}$.

Since, f(x) is odd function, so its graph is **symmetrical about opposite quadrants.** Also, we observe

$$\lim_{x\to 0^+} f(x) = +\infty \quad \text{and} \quad \lim_{x\to 0^-} f(x) = -\infty$$

and as $x \to \pm\infty \Rightarrow f(x) \to 0$.

Thus, $f(x) = \frac{1}{x}$ could be shown as in Fig.

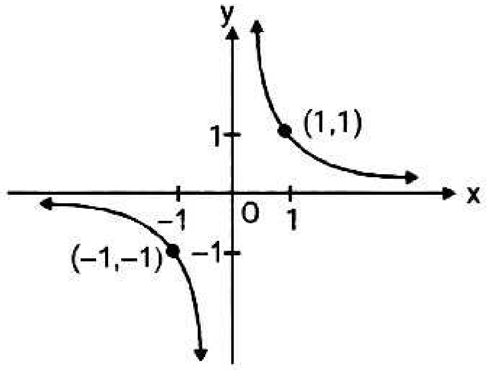


Fig.

(ii) Graph of $f(x) = \frac{1}{x^2}$

Here, $f(x) = \frac{1}{x^2}$ is an even function, so its graph is **symmetrical about y-axis.**

Domain of f(x) is $\mathbb{R} - \{0\}$ and range is $(0, \infty)$.

Also, as $y \to \infty$ as $\lim_{x\to 0^+} f(x)$ or $\lim_{x\to 0^-} f(x)$.

Thus, $f(x) = \frac{1}{x^2}$ could be shown as in Fig.

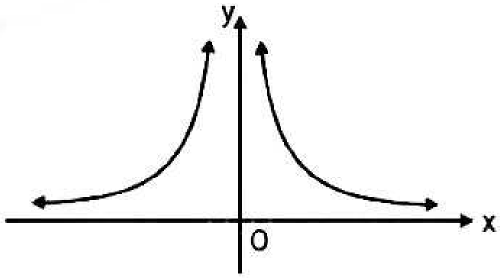


Fig.

(iii) Graph of $f(x) = \frac{1}{x^{2n-1}}$; $n \in N$

Here, $f(x) = \frac{1}{x^{2n-1}}$ is an odd function, so its graph is **Symmetrical in opposite quadrants.**

Also, $y \to \infty$ when $\lim_{x\to 0^+} f(x)$ and $y \to -\infty$ when $\lim_{x\to 0^-} f(x)$.

Thus, the graph for $f(x) = \frac{1}{x^3}$; $f(x) = \frac{1}{x^5}$, ..., etc. will be similar to the graph of $f(x) = \frac{1}{x}$ **which has asymptotes as coordinate axes,** shown as in Figure.

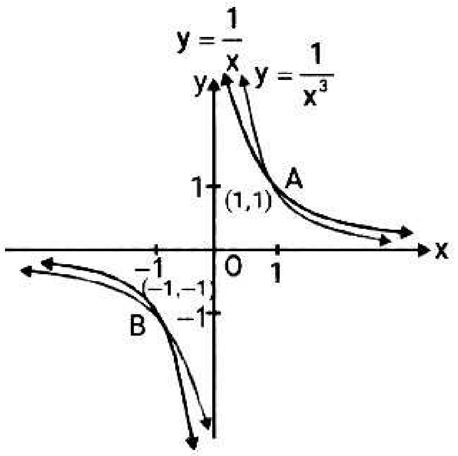


Fig.

(iv) Graph of $f(x) = \frac{1}{x^{2n}}$; $n \in N$

We observe that the function $f(x) = \frac{1}{x^{2n}}$ is an even function, so its graph is **symmetrical about y-axis.**

As, $y \to \infty$ as $\lim_{x \to 0^+} f(x)$ or $\lim_{x \to 0^-} f(x)$

and $y \to 0$ as $\lim_{x \to -\infty} f(x)$ or $\lim_{x \to +\infty} f(x)$.

The values of y decrease as the values of x increase. Thus, the graph of $f(x) = \frac{1}{x^2}$; $f(x) = \frac{1}{x^4}$, ..., etc. will be similar as the graph of $f(x) = \frac{1}{x^2}$, which has **asymptotes as coordinate axis.** Shown as in Fig.

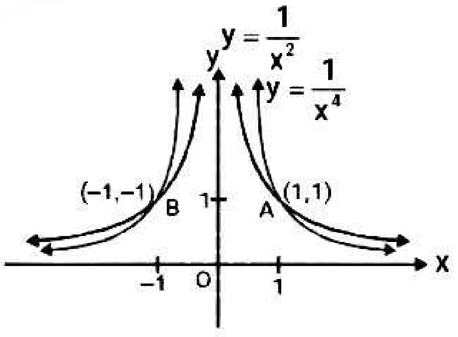


Fig.

1.3 Irrational Function

The algebraic function containing terms having non-integral rational powers of x are called irrational functions.

Graphs of Some Simple Irrational Functions

(i) Graph of $f(x) = x^{1/2}$

Here, $f(x) = \sqrt{x}$ is the portion of the parabola $y^2 = x$, which lies above x-axis.

Domain of $f(x) \in \mathbb{R}^+ \cup \{0\}$ or $[0, \infty)$

and range of $f(x) \in \mathbb{R}^+ \cup \{0\}$ or $[0, \infty)$

Thus, the graph of $f(x) = x^{1/2}$ is shown as;

Note : If $f(x) = x^n$ and $g(x) = x^{1/n}$, then f(x) and g(x) are inverse of each other.

$\therefore$ $f(x) = x^n$ and $g(x) = x^{1/n}$ is the mirror image about $y = x$.

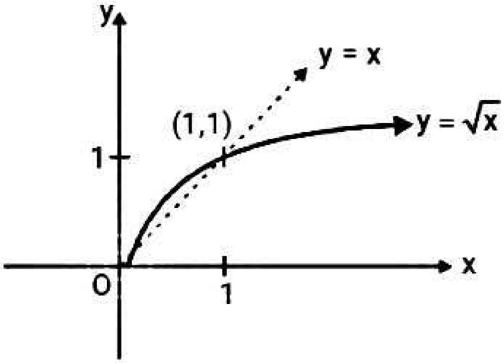


Fig.

(ii) Graph of $f(x) = x^{1/3}$

As discussed above, if $g(x) = x^3$. Then $f(x) = x^{1/3}$ is image of g(x) about $y = x$.

where domain $f(x) \in \mathbb{R}$.

and range of $f(x) \in \mathbb{R}$.

Thus, the graph of $f(x) = x^{1/3}$ is shown in Fig.

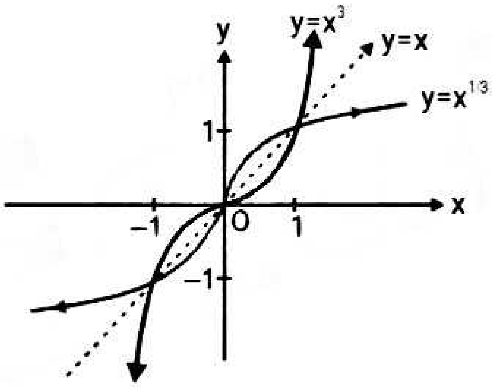


Fig.

(iii) Graph of $f(x) = x^{1/2n}$; $n \in N$

Here $f(x) = x^{1/2n}$ is defined for all $x \in [0, \infty)$ and the values taken by f(x) are positive.

So, domain and range of f(x) are $[0, \infty)$.

Here, the graph of $f(x) = x^{1/2n}$ is the mirror image of the graph of $f(x) = x^{2n}$ about the line $y = x$, when $x \in [0, \infty)$.

Thus, $f(x) = x^{1/2}$, $f(x) = x^{1/4}$, ... are shown as;

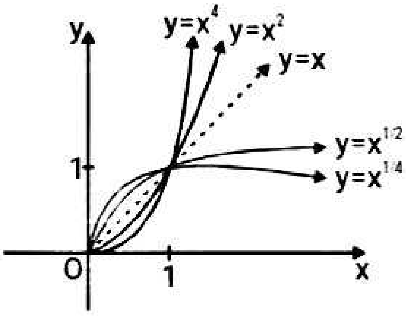


Fig.

(iv) Graph of $f(x) = x^{1/2n-1}$, when $n \in N$

Here, $f(x) = x^{1/2n-1}$ is defined for all $x \in \mathbb{R}$. So, domain of $f(x) \in \mathbb{R}$, and range of $f(x) \in \mathbb{R}$. Also the graph of $f(x) = x^{1/2n-1}$ is the mirror image of the graph of $f(x) = x^{2n-1}$ about the line $y = x$ when $x \in \mathbb{R}$.

Thus, $f(x) = x^{1/3}$, $f(x) = x^{1/5}$, ..., are shown as;

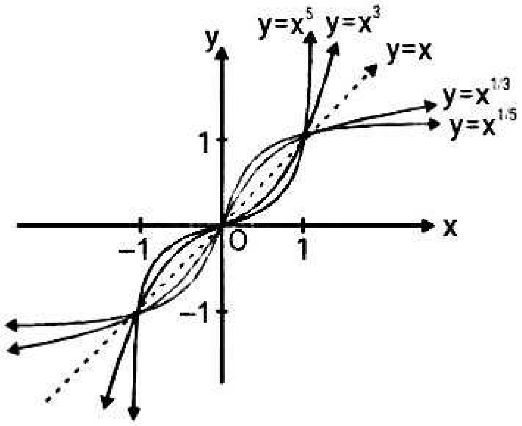


Fig.

1.4 Piecewise Functions

As discussed piecewise functions are:

(a) Absolute value function (or modulus function)

(b) Signum function

(c) Greatest integer function

(d) Fractional part function

(e) Least integer function

(i) Absolute value function (or modulus function)

$$y = |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

"It is the numerical value of x".

"It is symmetric about y-axis" where domain $\in \mathbb{R}$

and range $\in [0, \infty)$.

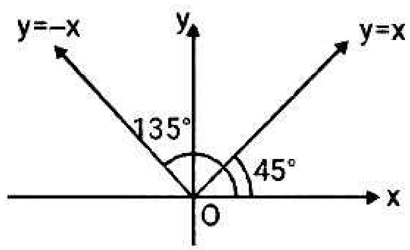


Fig.

Properties of modulus functions

(i) $|x| \leq a \Rightarrow -a \leq x \leq a$; $(a \geq 0)$

(ii) $|x| \geq a \Rightarrow x \leq -a$ or $x \geq a$; $(a \geq 0)$

(iii) $|x \pm y| \leq |x| + |y|$

(iv) $|x \pm y| \geq ||x| - |y||$

(ii) Signum function; y = Sgn(x)

It is defined by;

$$y = \text{Sgn}(x) = \begin{cases} \dfrac{|x|}{x} \text{ or } \dfrac{x}{|x|}; & x \neq 0 \\ 0; & x = 0. \end{cases} = \begin{cases} +1, & \text{if } x > 0 \\ -1, & \text{if } x < 0 \\ 0, & \text{if } x = 0 \end{cases}$$

Here, Domain of f(x) $\in \mathbb{R}$.

and Range of f(x) $\in \{-1, 0, 1\}$.

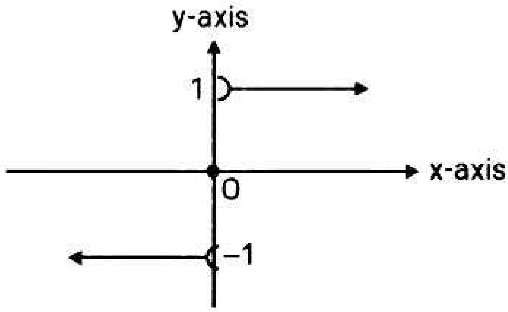


Fig.

(c) Greatest integer function

[x] indicates the integral part of x which is nearest and smaller integer to x. It is also known as **floor of x**.

Thus, [2, 3] = 2, [0.23] = 0, [2] = 2, [–8.0725] = –9, ...

In general; $n \leq x < n + 1$ ($n \in$ Integer) $\Rightarrow$ [x] = n.

Here, f(x) = [x] could be expressed graphically as;

x	[x]
$x \leq x < 1$	0
$1 \leq x < 2$	1
$2 \leq x < 3$	2

Thus, f(x) = [x] could be shown as;

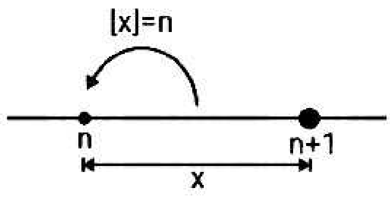


Fig.

Properties of greatest integer function

(i) [x] = x holds, if x is integer.

(ii) [x + I] = [x] + I, if I is integer.

(iii) $[x + y] \geq [x] + [y]$.

(iv) If $[\phi(x)] \geq I$, then $\phi(x) \geq I$.

(v) If $[\phi(x)] \leq I$, then $\phi(x) < I + 1$.

(vi) [–x] = –[x], if $x \in$ integer.

(vii) [–x] = –[x] –1, if $x \notin$ integer.

"It is also known as **stepwise function/floor of x.**"

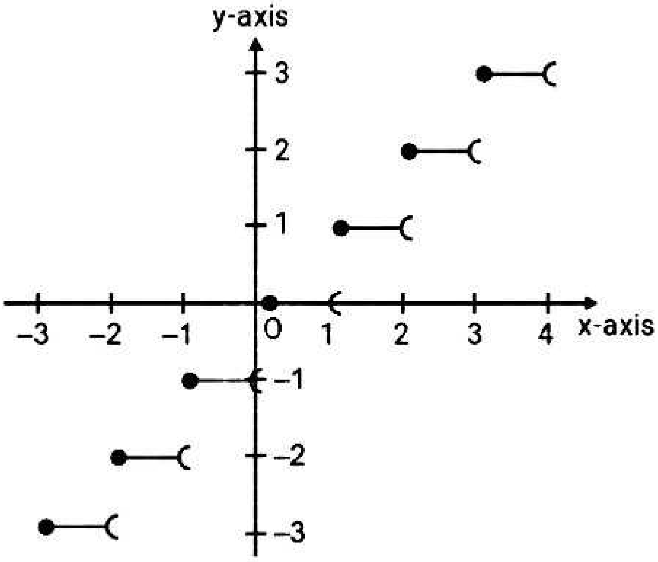


Fig.

(d) Fractional part of function

Here, {.} denotes the fractional part of x. Thus, in y = {x}.

$x = [x] + \{x\} = I + f$; where $I = [x]$ and $f = \{x\}$

$\therefore$ $y = \{x\} = x - [x]$, where $0 \le \{x\} < 1$; shown as:

x	{x}
$0 \le x < 1$	x
$1 \le x < 2$	$x - 1$
$2 \le x < 3$	$x - 2$
$-1 \le x < 0$	$x + 1$
$-2 \le x < -1$	$x + 2$

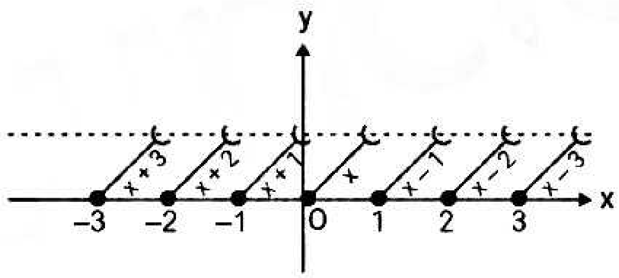


Fig.

Properties of fractional part of x

(i) $\{x\} = x$; if $0 \le x < 1$

(ii) $\{x\} = 0$; if $x \in$ integer.

(iii) $\{-x\} = 1 - \{x\}$; if $x \in$ integer.

(e) Least Integer Function

$y = (x) = \lceil x \rceil$,

(x) or $\lceil x \rceil$ indicates the integral part of x which is nearest and greatest integer to x.

It is known as ceiling of x.

Thus, $\lceil 2.3023 \rceil = 3$, $(0.23) = 1$, $(-8.0725) = -8$, $(-0.6) = 0$

In general, $n < x \le n + 1$ (n $\in$ integral))

i.e., $\lceil x \rceil$ or $(x) = n + 1$

shown as;

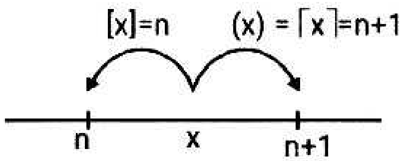


Fig.

Here, f(x) = (x) = $\lceil x \rceil$, can be expressed graphically as:

x	$\lceil x \rceil$ = (x)
$-1 < x \le 0$	0
$0 < x \le 1$	1
$1 < x \le 2$	2
$-2 < x \le -1$	-1
$-3 < x \le -2$	-2

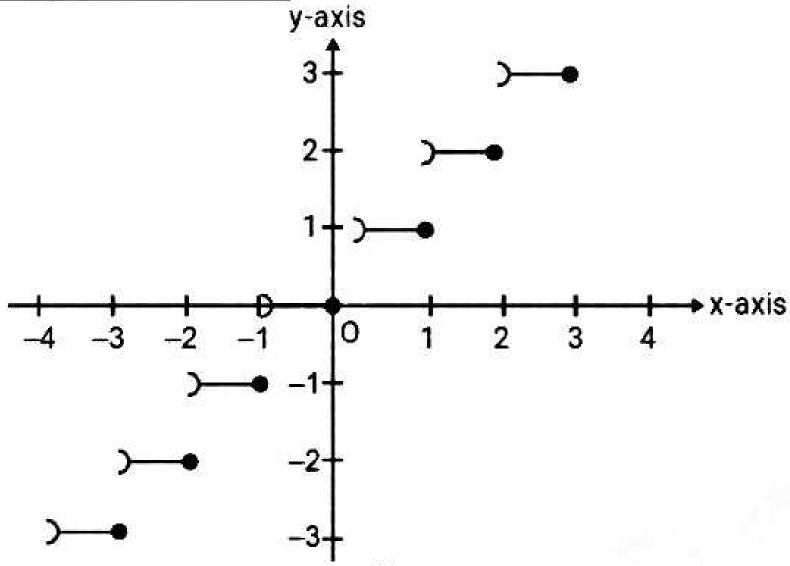


Fig.

Properties of least integer function

(i) (x) = $\lceil x \rceil$ = x, if

(ii) (x + I) = $\lceil x + I \rceil$ = (x) = I ; if I $\in$ integer.

(iii) Greatest integer converts x = I + f to [x] = I while $\lceil x \rceil$ converts to (I + 1).

2. Transcendental Functions

2.1 Trigonometric Function

(a) Sine Function : Here, f(x) = sin x can be discussed in two ways i.e., Graph diagram and Circle diagram where Domain of sine function is "$\mathbb{R}$" and range is [–1, 1].

Graph Diagram

(On x-axis and y-axis)

f(x) = sin x, increases strictly from –1 to 1 as x increases from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$, decreases strictly from 1 to –1 as x increases from $\frac{\pi}{2}$ to $\frac{3\pi}{2}$ and so on. We have graph as;

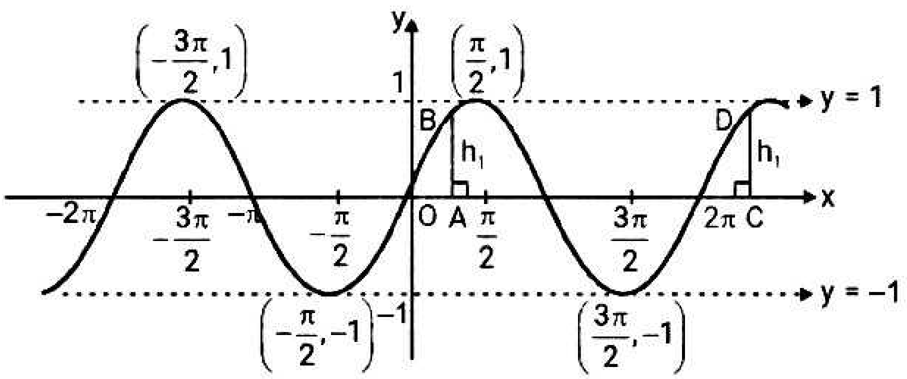


Fig.

Here, the height is same after every interval of 2π. (i.e., In above figure, AB = CD after every interval of 2π).

$\therefore$ sin x is called periodic function with period 2π.

Circle Diagram

(On trigonometric plane or using quadrants). Let a circle of radius '1', i.e., unit circle.

Then, $\sin \alpha = \frac{a}{1}$,

$\sin \beta = \frac{b}{1}$,

$\sin \gamma = -\frac{c}{1}$,

$\sin \delta = -\frac{d}{1}$, ... , shown as.

$\therefore$ sin x generates a circle of radius '1'.

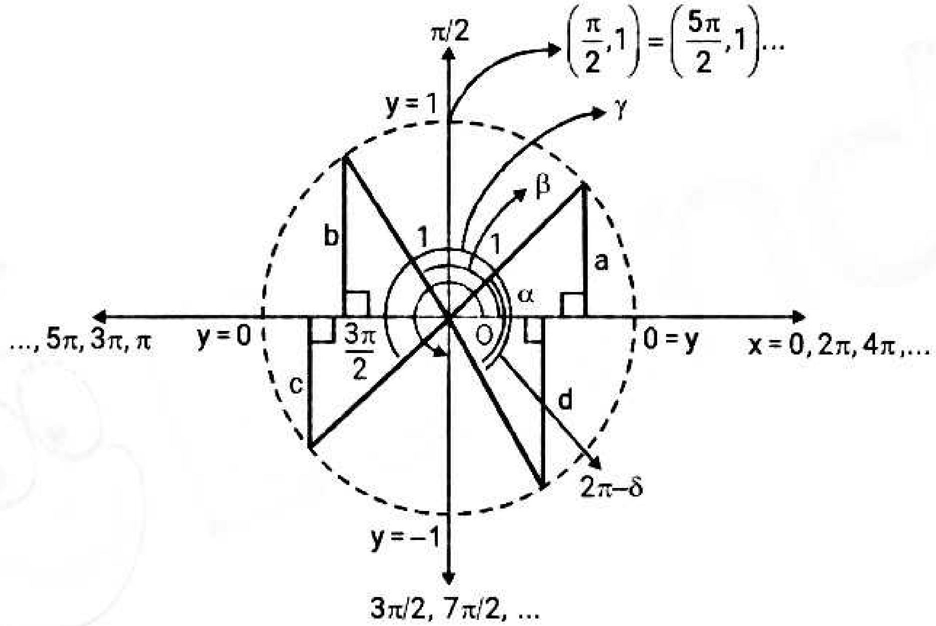


Fig.

(b) Cosine Function : Here, f(x) = cos x

The domain of cosine function is $\mathbb{R}$ and the range is [–1, 1].

Graph diagram (on x-axis and y-axis)

As discussed, cos x decreases strictly from 1 to –1 as x increases from 0 to π, increases strictly from –1 to 1 as x increases from π to 2π and so on. Also, cos x is periodic with period 2π.

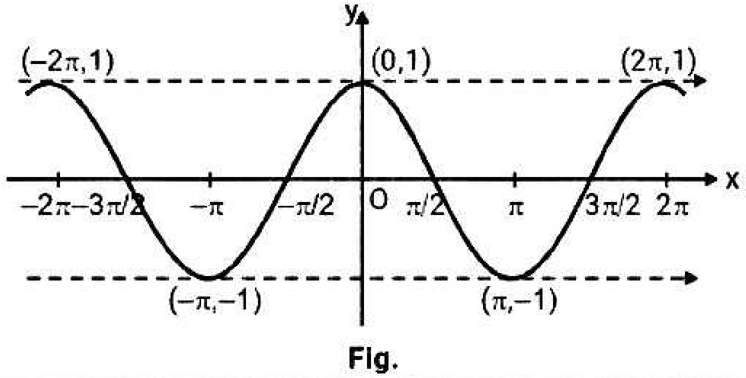


Fig.

Circle Diagram

Let a circle of radius '1', i.e., a unit circle.

Then, $\cos\alpha = \frac{a}{1}$, $\cos\beta = -\frac{b}{1}$,

$\cos\gamma = \frac{c}{1}$, $\cos\delta = \frac{d}{1}$

$\therefore$ cos x generates a circle of **radius '1'**.

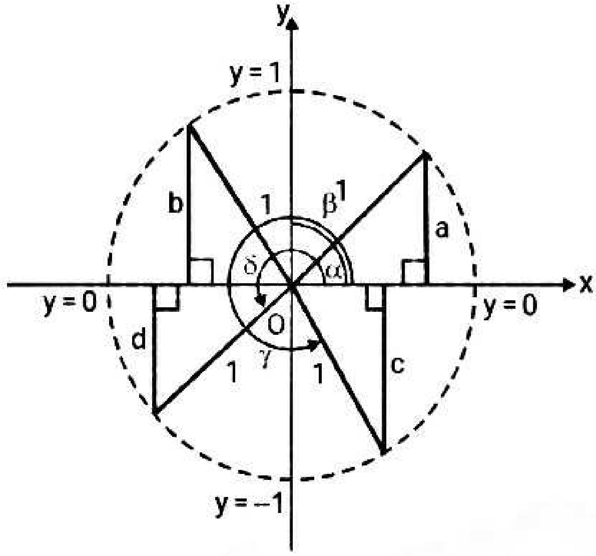


Fig.

(c) Tangent Function : f(x) = tan x

The domain of the function y = tan x is;

$$R = \left\{\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots\right\}$$

i.e., $\mathbb{R} - \left\{(2n+1)\frac{\pi}{2}\right\}$ and Range $\in \mathbb{R}$ or $(-\infty, \infty)$

The function y = tan x increases strictly from $-\infty$ to $+\infty$ as x increases from

$-\frac{\pi}{2}$ to $\frac{\pi}{2}$, $\frac{\pi}{2}$ to $\frac{3\pi}{2}$, $\frac{3\pi}{2}$ to $\frac{5\pi}{2}$, ... and so on.

The graph is shown as :

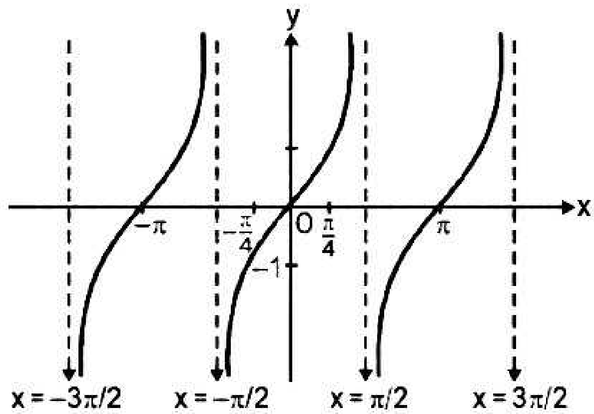


Fig.

Note : Here, the curve tends to meet at $x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2} + \frac{5\pi}{2}, \ldots$ but never meets or tends to infinity.

$\therefore \quad x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2} \ldots$ are asymptotes to $y = \tan x$.

(d) Cosecant Function

$f(x) = \text{cosec } x$

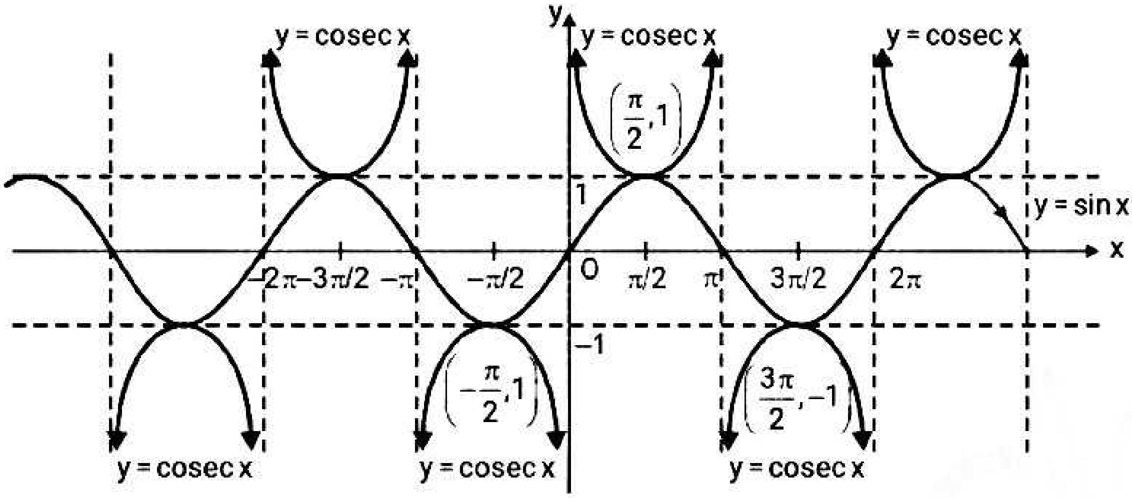


Fig.

Here, domain of $y = \text{cosec } x$ is,

$\mathbf{R} - \{0, \pm\pi, \pm2\pi, \pm3\pi, \ldots\}$

i.e., $\mathbf{R} - \{n\pi \mid n \in z\}$ and range $\in \mathbb{R} - (-1, 1)$

as shown in Fig.

The function $y = \text{cosec } x$ is periodic with period 2π.

(e) Secant Function : $f(x) = \sec x$

Here, domain $\in \mathbb{R} - \left\{(2n+1)\frac{\pi}{2}\middle| n \in z\right\}$

Range $\in \mathbb{R} - (-1, 1)$

Shown as:

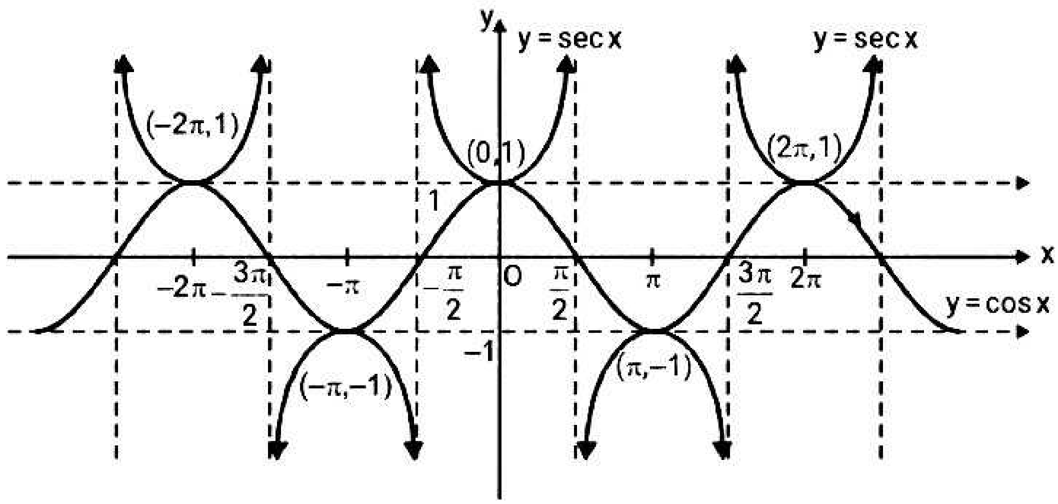


Fig.

The function y = sec x is periodic with period 2π.

Note :

(i) The curve y = cosec x tends to meet at $x = 0, \pm\pi, \pm 2\pi, \ldots$ at infinity.

$\therefore \quad x = 0, \pm\pi, \pm 2\pi, \ldots$

or $\quad x = n\pi,\ n \in$ integer are asymptote to y = cosec x.

(ii) The curve y = sec x tends to meet at $x = \pm\frac{\pi}{2} \pm \frac{3\pi}{2}, \ldots$ at infinity.

$\therefore x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots$ or $x = (2n+1)\frac{\pi}{2}$, $n \in$ integer are asymptote to y = cosec x.

(f) Cotangent Function

f(x) cos x

Here, domain $\in \mathbb{R} - \{n\pi \mid n \in z\}$ Range $\in \mathbb{R}$.

which is periodic with period π, and has $x = n\pi$, $n \in z$ as asymptotes. As shown in Fig.

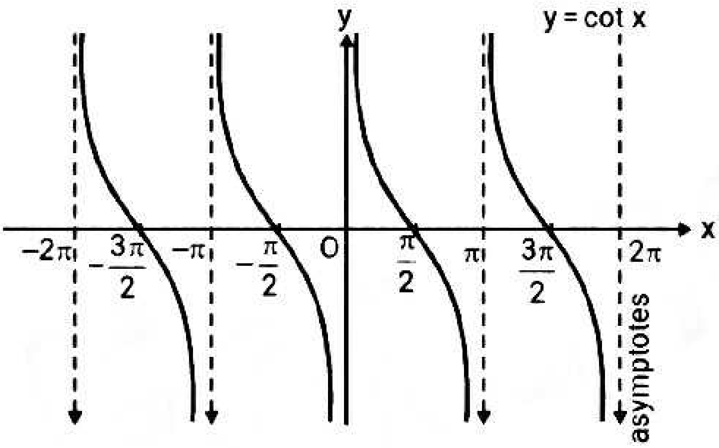


Fig.

2.2 Exponential Function

Here, $f(x) = a^x$, $a > 0$, $a \neq 1$, and $x \in \mathbb{R}$, where domain $\in \mathbb{R}$,

Range $\in (0, \infty)$.

Case I. a > 1

Here, $f(x) = y = a^x$ increase with the increase in x, i.e., f(x) is increasing function on $\mathbb{R}$.

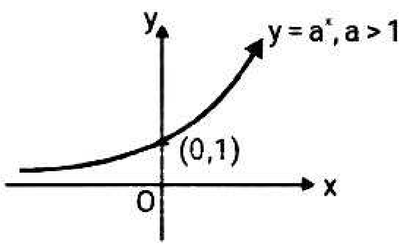


Fig.

For Ex;

$y = 2^x, y = 3^x, y = 4^x, \ldots$ have;

$2^x < 3^x < 4^x < \ldots$ for $x > 1$

and $\quad 2^x > 3^x > 4^x > \ldots$ for $0 < x < 1$.

and they can be shown as;

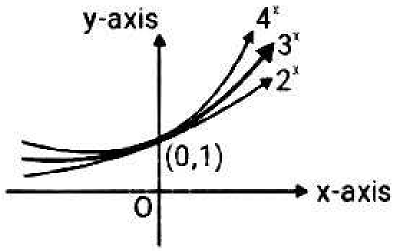


Fig.

Case II. $0 < a < 1$

Here, $f(x) = a^x$ decrease with the increase in x, i.e., f(x) is decreasing function on $\mathbb{R}$.

"In general, exponential function increases or decreases as ($a > 1$) or ($0 < a < 1$) respectively".

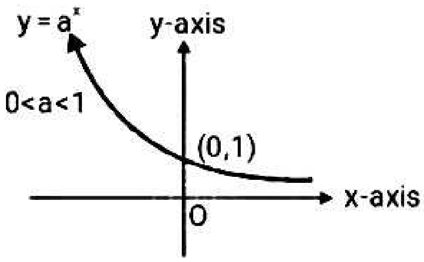


Fig.

2.3 Logarithmic Function

(Inverse of Exponential)

The function $f(x) = \log_a x$; ($x, a > 0$) and $a \neq 1$ is a logarithmic function.

Thus, the domain of logarithmic function is all real positive numbers and their range is the set $\mathbb{R}$ of all real numbers.

We have seen that $y = a^x$ is strictly increasing when $a > 1$ and strictly decreasing when $0 < a < 1$.

Thus, the function is invertible. The inverse of this function is denoted by $\log_a x$, we write

$$y = a^x \quad \Rightarrow \quad x = \log_a y;$$

where $x \in \mathbb{R}$ and $y \in (0, \infty)$ writing $y = \log_a x$ in place of $x = \log_a y$, we have the graph of $y = \log_a x$.

Thus, logarithmic function is also known as inverse of exponential function.

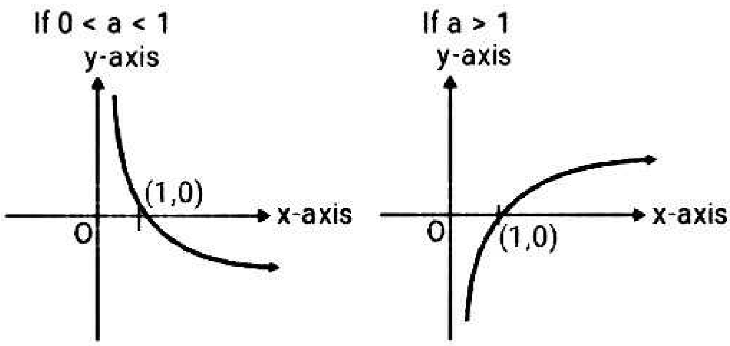


Fig.

Properties of logarithmic function

1. $\log_e (ab) = \log_e a + \log_e b$ $\{a, b > 0\}$

2. $\log_e \left(\frac{a}{b}\right) = \log_e a - \log_e b$ $\{a, b > 0\}$

3. $\log_e a^m = m \log_e a$ $\{a > 0 \text{ and } m \in \mathbb{R}\}$

4. $\log_a a = 1$ $\{a > 0 \text{ and } a \neq 1\}$

5. $\log_{b^m} a = \frac{1}{m}\log_b a$ $\{a, b > 0, b \neq 1 \text{ and } m \in \mathbb{R}\}$

6. $\log_b a = \frac{1}{\log_a b}$ $\{a, b > 0 \text{ and } a, b \neq 1\}$

7. $\log_b a = \frac{\log_m a}{\log_m b}$ $\{a, b > 0 \neq \{1\} \text{ and } m > 0\}$

8. $a^{\log_a m} = m$ $\{a, m > 0 \text{ and } a \neq 1\}$

9. $a^{\log_c b} = b^{\log_c a}$ $\{a, b, c > 0 \text{ and } c \neq 1\}$

10. If $\log_m x > \log_m y \Rightarrow \begin{cases} x > y, & \text{if } m > 1 \\ x < y, & \text{if } 0 < m < 1 \end{cases}$

which could be graphically shown as;

If m > 1 (Graph of $\log_m$ a)

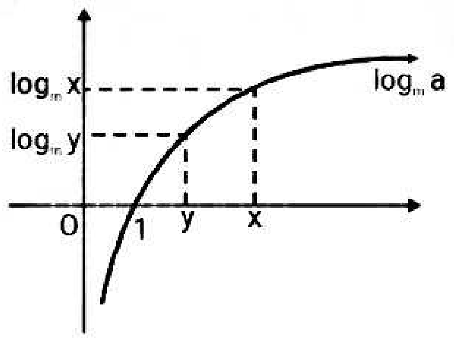


Fig.

$\Rightarrow \log_m x > \log_m y$ when $x > y$ and $m > 1$.

Again **if 0 < m < 1. (Graph of $\log_m$ a)**

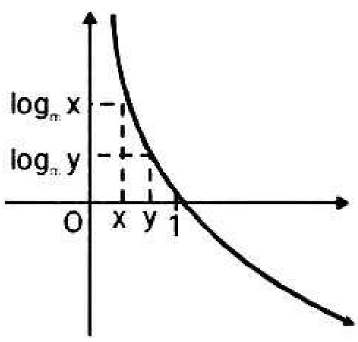


Fig.

$\Rightarrow \log_m x > \log_m y$; when $x < y$ and $m < m < 1$.

11. $\log_m a = b \Rightarrow a = m^b$ $\{a, m > 0;\ m \neq 1;\ b \in \mathbb{R}\}$

12. $\log_m a > b \Rightarrow \begin{cases} a > m^b; & \text{if } m > 1 \\ a < m^b; & \text{if } 0 < m < 1 \end{cases}$

13. $\log_m a < b \Rightarrow \begin{cases} a < m^b; & \text{if } m > 1 \\ a > m^b; & \text{if } 0 < m < 1 \end{cases}$

2.4 Geometrical Curves

(a) Straight line : $ax + by + c = 0$ (represents general equation of straight line). We know,

$$y = -\frac{c}{b} \quad \text{where } x = 0$$

and $$x = -\frac{c}{a} \quad \text{where } y = 0$$

joining above points we get required straight line.

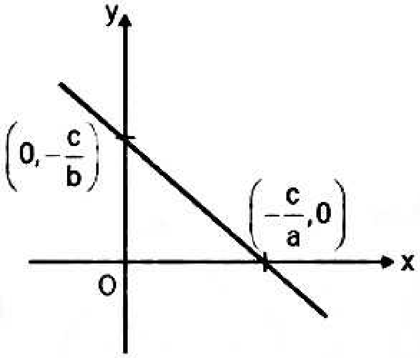


Fig.

(b) Circle : We know,

(i) $x^2 + y^2 = a^2$ is circle with centre (0, 0) and radius r.

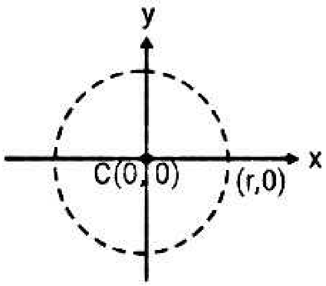


Fig.

(ii) $(x - a)^2 + (y - b)^2 = r^2$, circle with centre (a, b) and radius r.

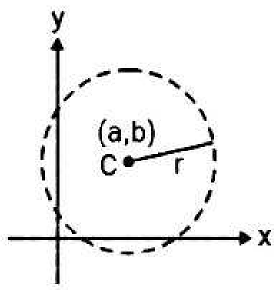


Fig.

(iii) $x^2 + y^2 + 2gr + 2fy + c = 0$; centre $(-g, -f)$; radius $\sqrt{g^2 + f^2 - c}$.

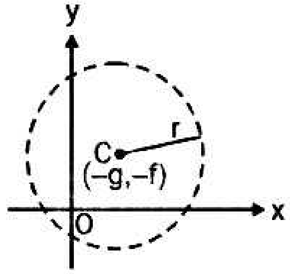


Fig.

(iv) $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$;
End points of diameter are (x_1, y_1) and (x_2, y_2).

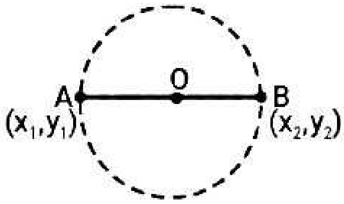


Fig.

(c) Parabola

(i) $y^2 = 4ax$
Vertex : (0, 0)
Focus : (a, 0)
Axis : x-axis or y = 0
Directrix : x = –a

(ii) $y^2 = -4ax$
Vertex : (0, 0)
Focus : (–a, 0)
Axis : x-axis or y = 0
Directrix : a = 0

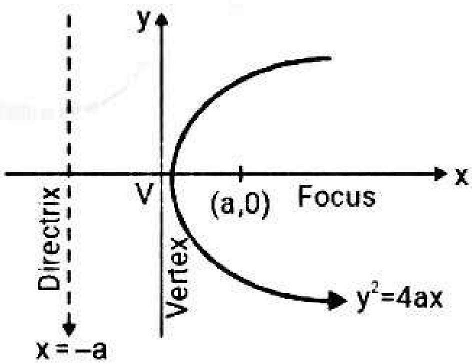


Fig.

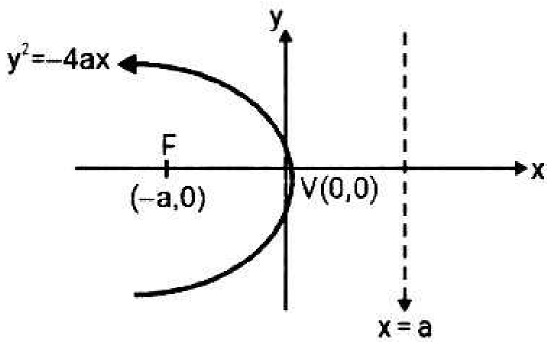


Fig.

(iii) $x^2 = 4ay$
Vertex : (0, 0)
Focus : (0, a)
Axis : y-axis or x = 0
Directrix : y = –a

(iv) $x^2 = -4ay$
Vertex : (0, 0)
Focus : (0, –a)
Axis : y-axis or x = 0
Directrix : y = 0

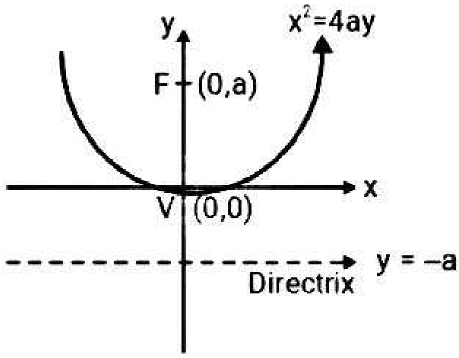


Fig.

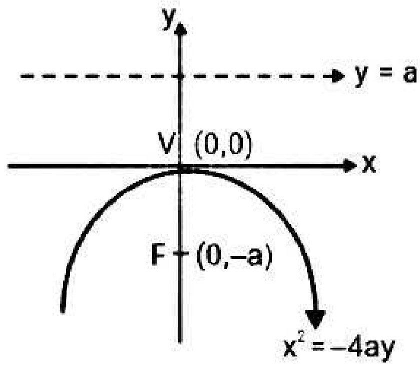


Fig.

(v) $(y - k)^2 = 4a(x - h)$

Vertex : (h, k)

Focus : (h + a, k)

Axis : x = h

Directrix : x = h – a

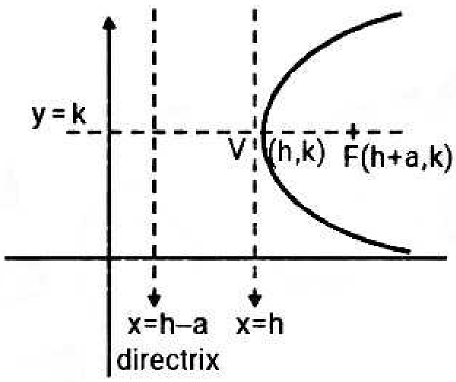


Fig.

(d) Ellipse

(i) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \; (a^2 > b^2)$

Centre : (0, 0)

Focus : (±ae, 0)

Vertex : (±a, 0)

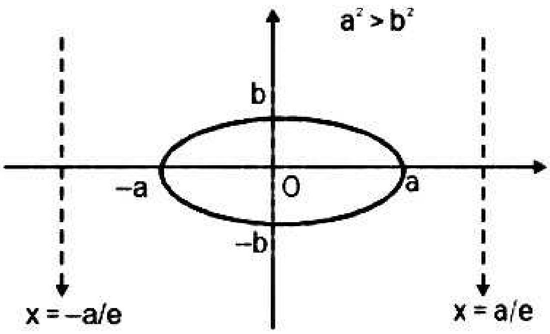


Fig.

Eccentricity : $e = \sqrt{1-\frac{b^2}{a^2}}$

Directrix : $x = \pm\frac{a}{e}$

(ii) $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1 \ (a^2 < b^2)$

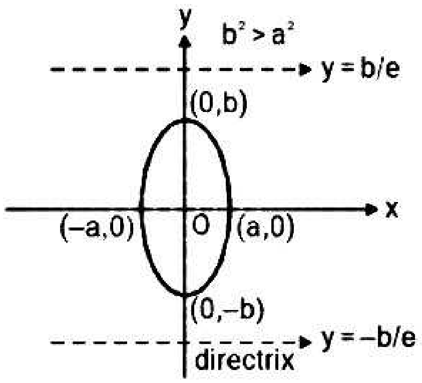


Fig.

(iii) $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2} = 1, \ (a^2 > b^2)$

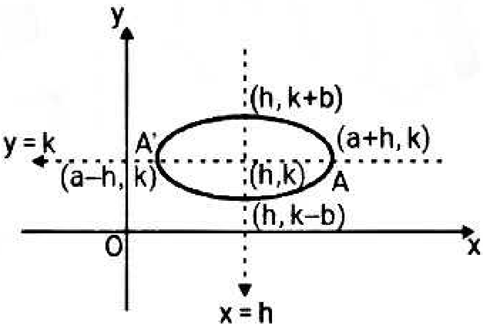


Fig.

(e) Hyperbola

(i) $\frac{x^2}{a^2}-\frac{y^2}{b^2} = 1$

Centre : (0, 0)

Focus : ($\pm$ae, 0)

Vertices : ($\pm$a, 0)

Eccentricity : $e = \sqrt{1+\frac{b^2}{a^2}}$

Directrix : $x = \pm\frac{a}{e}$

In above figure asymptotes are $y = \pm\frac{b}{a}x$.

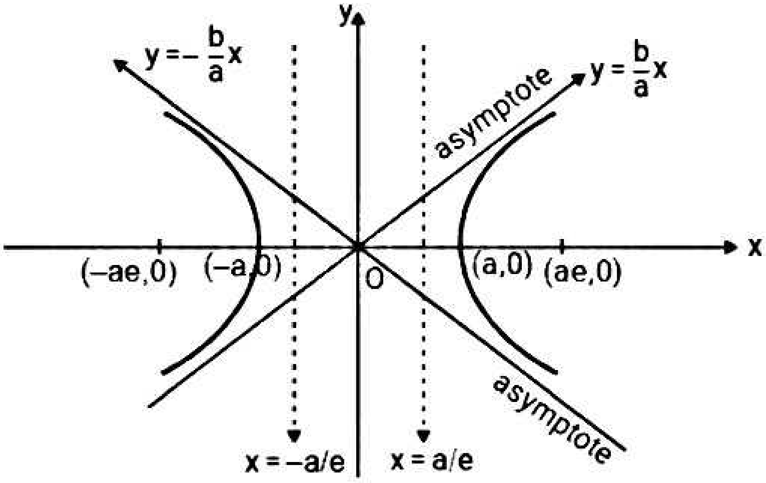


Fig.

(ii) $-\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

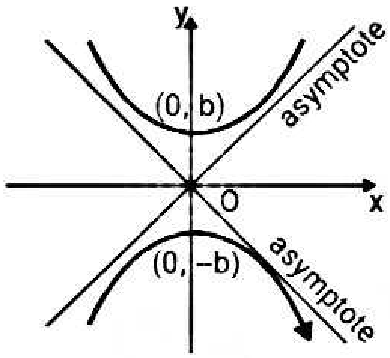


Fig.

(iii) $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$

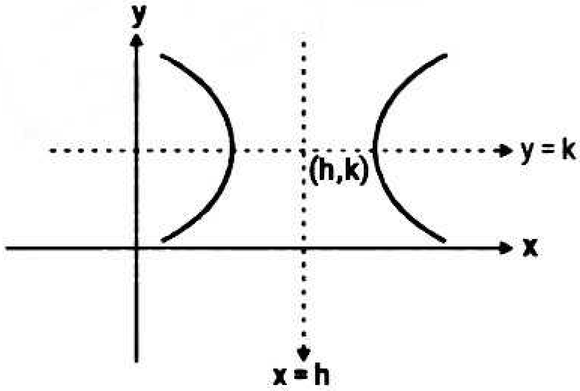


Fig.

(iv) $x^2 - y^2 = a^2$ **(Rectangular hyperbola)**

As asymptotes are perpendicular. Therefore, called rectangular hyperbola.

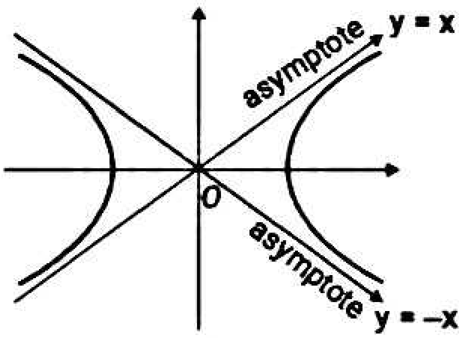


Fig.

(v) $xy = c^2$

Here, the asymptotes are x-axis and y-axis.

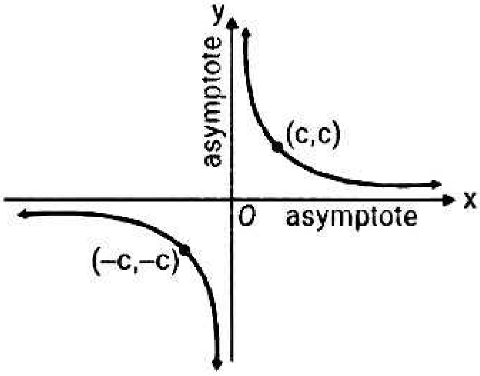


Fig.

2.5 Inverse Trigonometric Curves

As we know **trigonometric functions** are **many one** in their domain, **hence, they are not invertible.**

But their inverse can be obtained by restricting the domain so as to make invertible.

Note : Every inverse trigonometric is been converted to a function by shortening the domain.

For Ex : Let f(x) = sin x

We know, sin x is not invertible for $x \in \mathbb{R}$.

In order to get the inverse we have to define domain as:

$$x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$\therefore$ If $f : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to [-1, 1]$ defined by f(x) = sin x is invertible and inverse can be represented by:

$$y = \sin^{-1} x. \left(-\frac{\pi}{2} \le \sin^{-1} x \le \frac{\pi}{2}\right)$$

Similarly,

y = cos x becomes invertible when $f : [0, \pi] \to [-1, 1]$

y = tan x ; becomes invertible when $f : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to (-\infty, \infty)$

y = cot x ; becomes invertible when $f : (0, \pi) \to (-\infty, \infty)$

y = sec x ; becomes invertible when $f : [0, \pi] - \left\{\frac{\pi}{2}\right\} \to \mathbb{R}\ (-1, 1)$

y = cosec x; becomes invertible when $f : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\} \to \mathbb{R} - (-1, 1)$

(I) Graph of $y = \sin^{-1} x$;

where,

$x \in [-1, 1]$

and $\quad y = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

As the graph of $f^{-1}(x)$ is mirror image of f(x) about y = x.

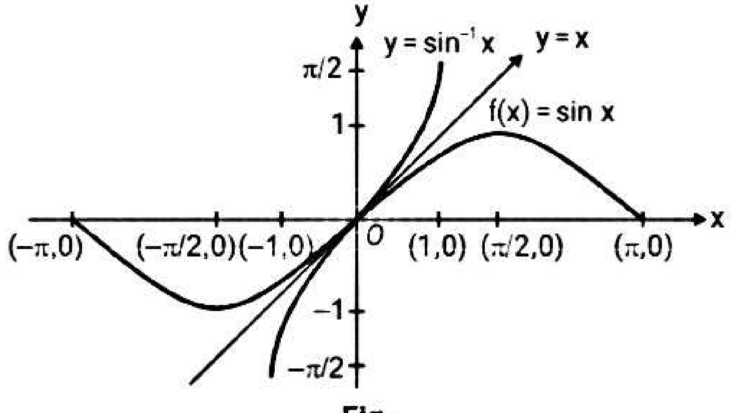


Fig.

(ii) Graph of $y = \cos^{-1} x$;

Here,

domain $\in [-1, 1]$

Range $\in [0, \pi]$

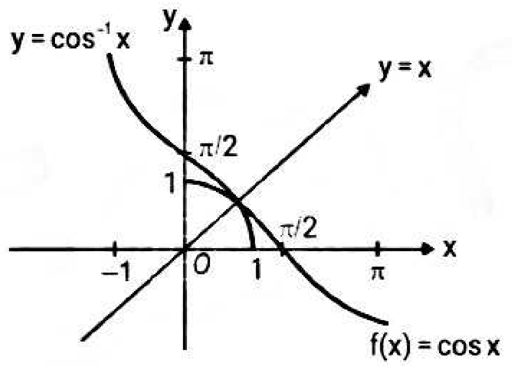


Fig.

(iii) Graph of $y = \tan^{-1} x$;

Here, domain $\in \mathbb{R}$, Range $\in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

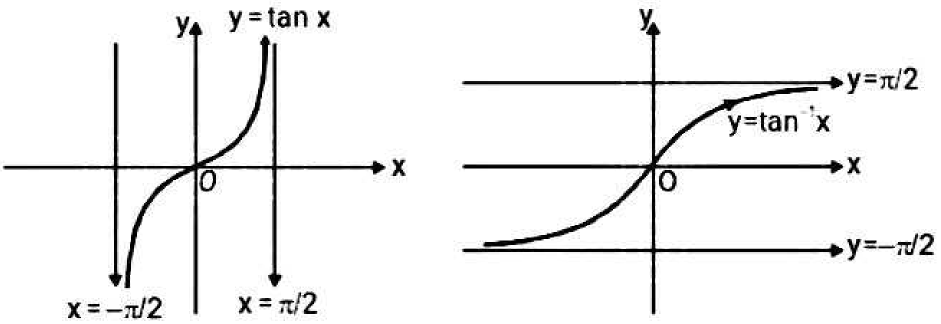


Fig.

As we have discussed earlier, "graph of inverse function is image of f(x) about y = x" or "by interchanging the coordinate axes".

(iv) Graph of $y = \cot^{-1} x$;

We know that the function $f : (0, \pi) \to \mathbb{R}$, given by $f(\theta) = \cot\theta$ is invertible.

$\therefore$ Thus, domain of $\cot^{-1} x \in \mathbb{R}$ and Range $\in (0, \pi)$.

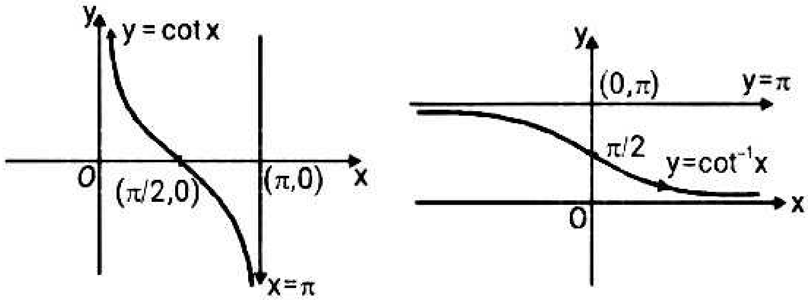


Fig.

(v) Graph for $y = \sec^{-1} x$;

The function $f : [0, \pi] - \left\{\frac{\pi}{2}\right\} \to (-\infty, -1] \cup [1, \infty)$ given by $f(\theta) = \sec\theta$ is invertible.

$\therefore$ $y = \sec^{-1} x$, has domain $\in \mathbb{R} - (-1, 1)$ and range $\in [0, \pi] - \left\{\frac{\pi}{2}\right\}$: shown as

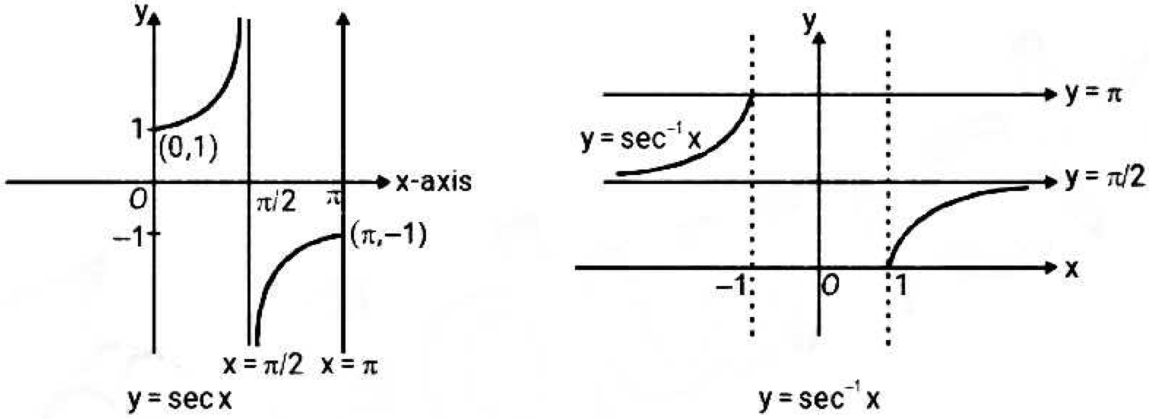


Fig.

(vi) Graph for $y = \mathrm{cosec}^{-1} x$;

As we know, $f : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\} \to \mathbb{R} - (-1, 1)$ is invertible given by $f(\theta) = \cos\theta$.

$\therefore$ $y = \mathrm{cosec}^{-1} x$; domain $\in \mathbb{R} - (-1, 1)$

Range $\in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$

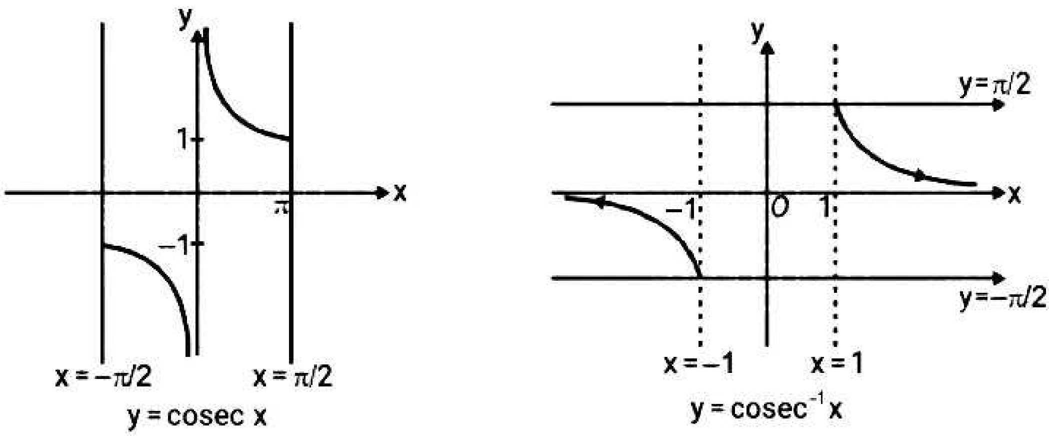


Fig.

Note : If no branch of an inverse trigonometric function is mentioned, then it means the principal value branch of that function.

In case no branch of an inverse trigonometric function is mentioned, it will mean the principal value branch of that function. (i.e.,)

	Function	Domain	Range	Principal Value Branch
1.	$\sin^{-1}x$	$[-1,1]$	$\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$	$-\frac{\pi}{2}\le y\le\frac{\pi}{2}$, where $y=\sin^{-1}x$
2.	$\cos^{-1}x$	$[-1,-1]$	$[0,\pi]$	$0\le y\le\pi$, where $y=\cos^{-1}x$
3.	$\tan^{-1}x$	R	$\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$	$-\frac{\pi}{2}<y<\frac{\pi}{2}$, where $y=\tan^{-1}x$
4.	$\operatorname{cosec}^{-1}x$	$(-\infty,1]\cup[1,\infty)$	$\left[-\frac{\pi}{2},\frac{\pi}{2}\right]-\{0\}$	$-\frac{\pi}{2}\le y\le\frac{\pi}{2}; y\neq 0$, where $y=\operatorname{cosec}^{-1}x$
5.	$\sec^{-1}x$	$(-\infty,1]\cup[1,\infty)$	$[0,\pi]-\left\{\frac{\pi}{2}\right\}$	$0\le y\le\pi$; $y\neq\left\{\frac{\pi}{2}\right\}$, where $y=\sec^{-1}x$
6.	$\cot^{-1}x$	R	$(0,\pi)$	$0<y<\pi$; where $y=\cot^{-1}x$.

2. BOUNDED FUNCTION

Definition : A function f : [a, b] → $\mathbb{R}$ is said to be **bounded** in the interval [a, b], if there exist two numbers k and K such that

$$k < f(x) < K \quad \forall \quad x \in [a, b].$$

or if its range is bounded.

Thus a function is **bounded** in [a, b], if it is both bounded above and bounded below in [a, b].

Ex :

1. $f(x) = x^2$ is bounded in [1, 2], since $1 \le f(x) \le 4 \ \forall \ x \in [1, 2]$.
2. $f(x) = \dfrac{1}{x}$ is bounded in (0, 1).
3. $f(x) = \dfrac{x}{x-1}$ is not bounded in [1, 4].

Results :

(a) Every bounded sequence has a limit points.

(b) A number p is a limit point of a sequence $\langle a_n \rangle$ iff there exists a subsequence $\langle a_{n_k} \rangle$ of $\langle a_n \rangle$ such that $a_{n_k} \to p$.

(c) Every closed interval [a, b] is a closed set.

(d) A closed set contains all its limit points.

(e) If p is a limit point of a set S and $S \subset T$, then p is also a limit point of T.

Unbounded function

If a function is neither bounded above nor below, then this is known as an **unbounded function**.

Ex :

1. f(x) = x is unbounded for $x \in (-\infty, \infty)$.
2. $f(x) = \dfrac{1}{x}$ is unbounded on any interval that includes x = 0.

Properties of bounded functions

1. If f,g are bounded, then f + g, f - g, f·g, and f/g are also bounded.

2. If f is bounded, then | f | is bounded and also the composition fog is bounded for any function g. In particular, any transformation of f is also bounded.

Note: If g is bounded, then (fog) need not be bounded. For instance, sin(x) is bounded, but cot(sin(x)) is not. (Indeed, values of sin(x) get arbitrarily close to 0, and cotangent tends to infinity for argument going to zero.)

3. If f is bounded and g does not approach 0 arbitrarily close (that is, inf(|g|) > 0, then f /g is bounded

4. A sum/difference of a bounded and an unbounded function is again unbounded.

Note: Adding/subtracting two unbounded functions may yield a bounded function. For product and division anything can happen.

Ex: A product of two unbounded functions may be bounded.

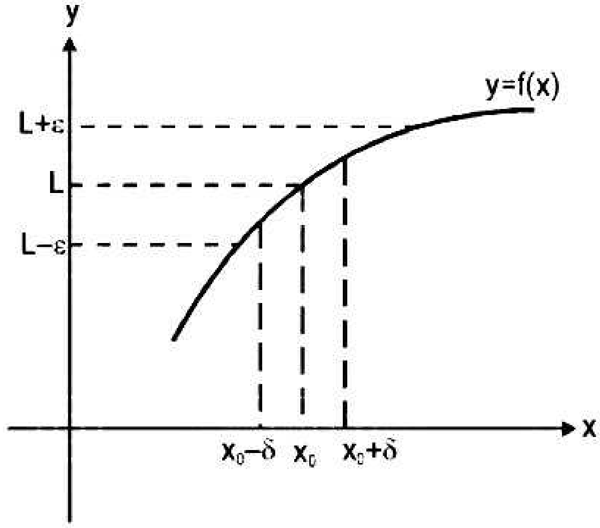
x·(1/x) = 1.

5. Composition of two unbounded functions may yield a bounded function.

For Ex, 1/x is unbounded, so is (e^x + 1), but when we substitute the latter into the former, we get 1/(e^x + 1), which is a bounded function. Indeed, inf(e^x + 1) = 1, so we are separated from zero, and 1 is a bounded function (constant), so the ratio is bounded..

3. LIMIT OF A FUNCTION OF ONE VARIABLE

If L is a real number, then $\lim_{x \to x_0} f(x) = L$ means that the value f(x) can be made as close to L as we wish by taking x sufficiently close to x_0. This is made precise in the following definition.



Figure

Definition : A function f is said to tend to a limit l as x tends to a, written as $\lim_{x \to a} f(x) = l$, if given any $\varepsilon > 0$ (however small), there exists some $\delta > 0$ (depending on ε) such that

$$| f(x) - l | < \varepsilon \quad \text{whenever} \quad 0 < | x - a | < \delta.$$

Figure depicts the graph of a function for which $\lim_{x \to x_0} f(x)$ exists.

Note : $\lim_{x \to a} f(x)$ should not be understood as substituting x = a in f(x). If fact, f may not be defined at x = a.

Ex : $\lim_{x \to 1} (2x + 3) = 5.$

Let $\varepsilon > 0$ be given. We have

$$| (2x + 3) - 5 | = | 2x - 2 | = 2 | x - 1 |.$$

Now $| (2x + 3) - 5 | < \varepsilon$ when $2 | x - 1 | < \varepsilon$ or $| x - 1 | < \frac{1}{2} \varepsilon$.

Choosing $\delta = \frac{1}{2}\varepsilon$, $| (2x + 3) - 5 | < \varepsilon$ when $| x - 1 | < \delta$.

Hence $\lim_{x \to 1} (2x + 3) = 5.$

Ex : If c and x are arbitrary real numbers and f(x) = cx, then

$$\lim_{x \to x_0} f(x) = c\,x_0.$$

To prove this, we write

$$|f(x) - cx_0| = |cx - cx_0| = |c||x - x_0|.$$

If $c \neq 0$, this yields

$$|f(x) - cx_0| < \varepsilon \qquad ....(*)$$

if

$$|x - x_0| < \delta,$$

where δ is any number such that $0 < \delta \leq \varepsilon/|c|$. If $c = 0$, then $f(x) - cx_0 = 0$ for all x, so (*) holds for all x.

One-Sided Limits

The function

$$f(x) = 2x \sin \sqrt{x}$$

satisfies the inequality

$$|f(x)| < \varepsilon$$

if $0 < x < \delta = \varepsilon/2$. However, this does not mean that $\lim_{x \to 0} f(x) = 0$, since f is not defined for negative x, as it must be to satisfy the conditions of Definition with $x_0 = 0$ and $L = 0$. The function

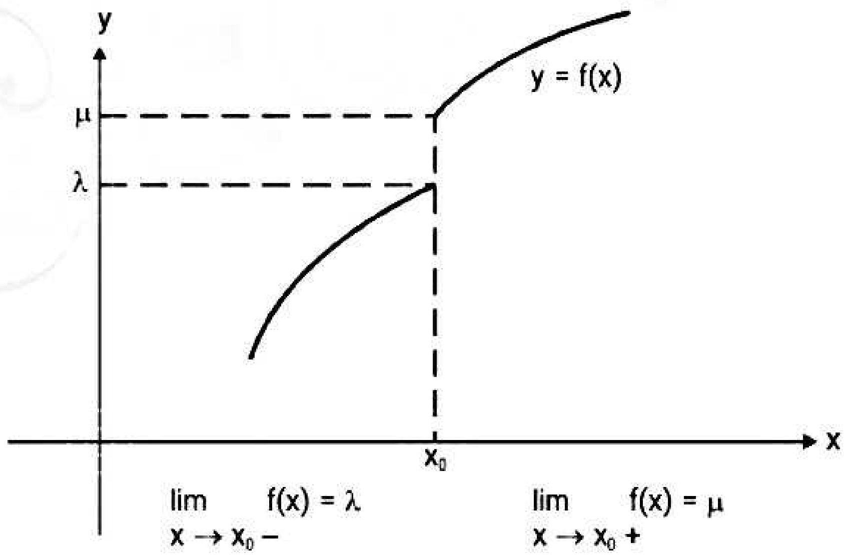
$$g(x) = x + \frac{|x|}{x}, \quad x \neq 0,$$

can be rewritten as

$$g(x) = \begin{cases} x + 1, & x > 0, \\ x - 1, & x < 0; \end{cases}$$

hence, every open interval containing $x_0 = 0$ also contains points x_1 and x_2 such that $|g(x_1) - g(x_2)|$ is as close to 2 as we places. Therefore, $\lim_{x \to x_0} g(x)$ does not exist.

Although f(x) and g(x) do not approach limits as x approaches zero, they each exhibit a definite sort of limiting behavior for all positive values of x, as does g(x) for small negative values of x. The kind of behavior we have in mind is defined precisely as follows.



Figure

Left Hand and Right Hand Limits

Definition : A function f is said to tend to a limit l as x tends to a from the left, if given any $\varepsilon > 0$ (however small), there exists a $\delta > 0$ (depending on ε) such that $| f(x) - l | < \varepsilon$, whenever $a - \delta < x < a$.

$$\lim_{x \to a} f(x) = l.$$

It is called the **left hand limit** at a

Definition : A function f is said to tend to a limit l as x tends to a from the **right,** if given any $\varepsilon > 0$ (however small), there exists a $\delta > 0$ (depending on ε) such that $|f(x) - l| < \varepsilon$, whenever $a < x < a + \delta$.

$$\lim_{x \to a^+} f(x) = l.$$

It is called the **right hand limit** at a

Remarks 1. $x \to a^-$ $\Rightarrow$ $x \to a$ through values less than a,

$x \to a^+$ $\Rightarrow$ $x \to a$ through values greater than a.

Remarks 2. Clearly, $\lim_{x \to a} f(x)$ exists if and only if $\lim_{x \to a-} f(x)$ and $\lim_{x \to a+} f(x)$ both exist and are equal.

Hence $\lim_{x \to a} f(x) = \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$.

Ex : $|x| = x$, if $x \geq 0$ and $|x| = -x$, if $x < 0$

$$\lim_{x \to 0-} f(x) = \lim_{\substack{x \to a \\ (x<0)}} \frac{|x|}{x} = \lim_{x \to 0} \frac{(-x)}{x} = -1.$$

$$\lim_{x \to 0+} f(x) = \lim_{\substack{x \to a \\ (x>0)}} \frac{|x|}{x} = \lim_{x \to 0} \frac{x}{x} = 1.$$

Hence $\lim_{x \to 0} \frac{|x|}{x}$ does not exist, since $\lim_{x \to 0-} f(x) \neq \mathrm{Lim}_{x \to 0+} f(x)$

Ex : Show that the function f, defined on $\mathbb{R}\backslash\{0\}$ by $f(x) = \sin(1/x)$, whenever $x \neq 0$, does not approach 0 as $x \to 0$.

Sol. Let $\varepsilon = \frac{1}{2} > 0$ and let δ be any positive number.

By Archimedean property, for $1/\delta > 0$, $\exists\, n \in N$ such that

$$2n\pi + \frac{\pi}{2} > \frac{1}{\delta} \text{ or } \frac{1}{2n\pi + \pi/2} < \delta.$$

If $x = \frac{1}{2n\pi + \pi/2}$, then $0 < |x - 0| < \delta$.

Now $|\sin(1/x) - 0| = |\sin(2n\pi + \pi/2) = 1 > \varepsilon$.

Thus there is an $\varepsilon = \frac{1}{2} > 0$, such that for each $\delta > 0$, $\exists$ some point $x = \frac{1}{2n\pi + \pi/2}$, for which $|\sin(1/x) - 0| > \varepsilon$ and $0 < |x - 0| < \delta$.

Hence $\lim_{x \to \infty} \sin(1/x) \neq 0$.

Ex : Let

$$g(x) = \frac{x + |x|(1+x)}{x} \sin\frac{1}{x}, \quad x \neq 0.$$

If $x < 0$, then

$$g(x) = -x \sin\frac{1}{x},$$

so

$$\lim_{x \to 0-} g(x) = 0,$$

since

$$|g(x) - 0| = \left|x \sin\frac{1}{x}\right| \le |x| < \varepsilon$$

if $-\varepsilon < x < 0$; that is, Definition is satisfied with $\delta = \varepsilon$. If $x > 0$, then

$$g(x) = (2 + x)\sin\frac{1}{x},$$

which takes on every values between –2 and 2 in every interval $(0, \delta)$. Hence, $g(x)$ does not approach a right-hand limit as x approaches 0 from the right. This shows that a function may have a limit from one side at a point but fail to have a limit from the other side.

Left and right-hand limit are also called *one-sided limits*.

$$\lim_{x \to x_0^-} f(x) = f(x_0^-) \quad \text{and} \quad \lim_{x \to x_0^+} f(x) = f(x_0^+).$$

Ex : Let

$$f(x) = 1 - \frac{1}{x^2}, \quad g(x) = \frac{2|x|}{1+x}, \quad \text{and} \quad h(x) = \sin x.$$

Then

$$\lim_{x \to \infty} f(x) = 1,$$

since

$$|f(x) - 1| = \frac{1}{x^2} < \varepsilon \quad \text{if} \quad x > \frac{1}{\sqrt{\varepsilon}},$$

and

$$\lim_{x \to \infty} g(x) = 2,$$

since

$$|g(x) - 2| = \left|\frac{2x}{1+x} - 2\right| = \frac{2}{1+x} < \frac{2}{x} < \varepsilon \quad \text{if} \quad x > \frac{2}{\varepsilon}.$$

However, $\lim_{x \to \infty} h(x)$ does not exist, since h assumes all values between –1 and 1.

Some Theorems on limit

Theorem : $\lim_{x \to a} f(x)$, if it exists, is unique.

Proof. Let $\lim_{x \to a} f(x) = l$, and $\lim_{x \to a} f(x) = l'$. ...(1)

We shall show that $l = l'$. Let $\varepsilon > 0$ be given.

Using (1), there exists $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$|f(x) - l| < \frac{1}{2}\varepsilon, \quad \text{where} \quad 0 < |x - a| < \delta_1, \qquad ...(2)$$

and

$$|f(x) - l'| < \frac{1}{2}\varepsilon, \quad \text{where} \quad 0 < |x - a| < \delta_2, \qquad ...(3)$$

Let $\delta = \min(\delta_1, \delta_2)$. Then, by (2) and (3),

$$|f(x) - l| < \frac{1}{2}\varepsilon, \text{ where } |x - a| < \delta,$$

$$|f(x) - l'| < \frac{1}{2}\varepsilon, \text{ when } |x - a| < \delta.$$

Now $$|l - l'| = |l - f(x) + f(x) - l'|$$

$$\leq |l - f(x)| + |f(x) - l'| < \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon$$

$\therefore$ $|l - l'| < \varepsilon$, when $0 < |x - a| < \delta$.

Since ε is arbitrarily small, $|l - l'| = 0$. Hence $l = l'$.

NOTE : Limit does not necessarily exist.

Theorem : If $\lim\limits_{x \to a} f(x) = l$ and $\lim\limits_{x \to a} g(x) = m$, then

(i) $\lim\limits_{x \to a} [f(x) \pm g(x)] = l \pm m.$ (ii) $\lim\limits_{x \to a} [f(x)\, g(x)] = lm.$

(iii) $\lim\limits_{x \to a} \left[\dfrac{f(x)}{g(x)}\right] = \dfrac{l}{m}; m \neq 0$

Proof.

(i) Let $\varepsilon > 0$ be given. There exist $\delta_1 > 0$, $\delta_2 > 0$ such that

$$|f(x) - l| < \frac{1}{2}\varepsilon, \quad \text{whenever} \quad 0 < |x - a| < \delta_1,$$

and $$|g(x) - m| < \frac{1}{2}\varepsilon, \quad \text{whenever} \quad 0 < |x - a| < \delta_2,$$

Let $\delta = \min(\delta_1, \delta_2)$. Then

$$|f(x) - l| < \frac{1}{2}\varepsilon, \quad \text{whenever} \quad 0 < |x - a| < \delta,$$

and $$|g(x) - m| < \frac{1}{2}\varepsilon, \quad \text{whenever} \quad 0 < |x - a| < \delta,$$

Now $|\{f(x) \pm g(x)\} - (l \pm m)| = |(f(x) - l| \pm (g(x) - m)|$

$\leq |f(x) - l| + |g(x) - m| < \varepsilon/2 + \varepsilon/2 = \varepsilon$, when $0 < |x - a| < \delta$

$\therefore$ $|\{f(x) \pm g(x)| - (l \pm m)| < \varepsilon$, whenever $0 < |x - a| < \delta$.

Hence $\lim\limits_{x \to a} [f(x) \pm g(x)] = l \pm m$.

(ii) We have $|f(x)\, g(x) - lm|$

$= |\, g(x)\, (f(x) - l) + l(g(x) - m)\, |$

$\leq |\, g(x)\, |\, |\, f((x) - l\, | + |\, l\, |\, |\, g(x) - m\, |.$ ...(1)

Since $\lim\limits_{x \to a} f(x) = l$ and $\lim\limits_{x \to a} g(x) = m$, for $\varepsilon' > 0$, there exists some $\delta > 0$ such that

$|\, g(x) - l\, | < \varepsilon'$ and $|\, g(x) - m\, | < \varepsilon'$, $0 < |\, x - a\, | < \delta$. ...(2)

Now $|\, g(x)\, | = |\, m + g(x) - m\, | \leq |\, m\, | + |\, g(x) - m\, |$

or $|\, g(x)\, | < |\, m\, | + \varepsilon'$, when $0 < |\, x - a\, | < \delta$. ...(3)

From (1), (2) and (3) ; we obtain

$$|f(x)\,g(x) - lm|$$

$$< (|m| + \varepsilon')\,\varepsilon' + |l|\,\varepsilon', \quad \text{when} \quad 0 < |x-a| < \delta$$

$$< (|m| + |l| + 1)\,\varepsilon', \quad \text{when} \quad 0 < |x-a| < \delta. \quad (\because\ \varepsilon' < 1)$$

Let us choose ε' such that

$$\varepsilon' < \frac{\varepsilon}{(|m|+|l|+1)},\ \varepsilon > 0.$$

$\therefore$ $|f(x)\,g(x) - lm| < \varepsilon$, whenever $0 < |x - a| < \delta$.

Hence $\lim\limits_{x\to a} [f(x)\,g(x)] = lm = \lim\limits_{x\to a} f(x) \,.\, \lim\limits_{x\to a} g(x)$.

(iii) Consider $\left|\dfrac{f(x)}{g(x)} - \dfrac{l}{m}\right| = \dfrac{|mf(x) - l\,g(x)|}{|g(x)|.|m|}$

$$= \frac{|mf(x) - lm + lm - l\,g(x)|}{|g(x)|.|m|}$$

$$= \frac{|m(f(x) - l) + l(m - g(x))|}{|g(x)|.|m|}$$

$$\le \frac{|m|.|f(x) - l| + |l||g(x) - m|}{|m|.|g(x)|} \qquad ...(1)$$

Since $\lim\limits_{x\to a} g(x) = m \neq 0$, so there exists a $\delta_1 > 0$ such that

$$|g(x)| > \frac{1}{2}|m|, \quad \text{when} \quad 0 < |x-a| < \delta_1$$

$$\Rightarrow \quad \frac{1}{|g(x)|} < \frac{2}{|m|}, \quad \text{when} \quad 0 < |x-a| < \delta_1 \qquad ...(2)$$

Since $\lim\limits_{x\to a} f(x) = l$, there exists some $\delta_2 > 0$ such that

$$|f(x) - l| < \frac{1}{4}\varepsilon\,|m|, \quad \text{when} \quad 0 < |x-a| < \delta_2. \qquad ...(3)$$

Since $\lim\limits_{x\to a} g(x) = m$, there exists some $\delta_3 > 0$ such that

$$|g(x) - m| < \frac{1}{4}\varepsilon\frac{|m|^2}{|l|}, \quad \text{when} \quad 0 < |x-a| < \delta_3 \qquad ...(4)$$

Let $\delta = \min(\delta_1, \delta_2, \delta_3)$. From (1), (2), (3), (4) ; we get

$$\left|\frac{f(g)}{g(x)} - \frac{l}{m}\right| < \frac{2}{|m|^2}\left[\frac{1}{4}\varepsilon|m|^2 + \frac{1}{4}\varepsilon|m|^2\right] = \varepsilon, \quad \text{when } 0 < |x-a| < \delta.$$

Hence $\lim\limits_{x\to a}\left[\dfrac{f(x)}{g(x)}\right] = \dfrac{l}{m}$, provided $m \neq 0$.

Ex. Let $f(x) = x^2 \sin(1/x)$, when $x \neq 0$, $f(0) = 0$, and $g(x) = x$.

Ex. Evaluate $\lim_{x \to 0} \frac{f(x)}{g(x)}$ and $\lim_{x \to 0} f(x)\, g(x)$.

Hint. $\lim_{x \to 0} \frac{f(x)}{g(x)} = \lim_{x \to 0} x \sin\frac{1}{x} = 0,$

$$\lim_{x \to 0} f(x)\, g(x) = \lim_{x \to 0} x^2 . \lim_{x \to 0} x \sin\frac{1}{x} = 0.$$

Theorem : If $\lim_{x \to a} f(x) = l \neq 0$, then there exists a $\delta > 0$ such that

$$|f(x)| > \frac{1}{2}|l|, \text{ where } 0 < |x - a| < \delta.$$

Proof. Since $l \neq 0$, so $|l| > 0$.

Let us choose $\varepsilon > 0$ such that $\varepsilon = \frac{1}{2}|l| > 0$.

Now $\lim_{x \to a} f(x) = l \Rightarrow$ for any $\varepsilon > 0$, $\exists$ a $\delta > 0$ such that

$$|f(x) - l| < \varepsilon, \quad \text{when} \quad 0 < |x - a| < \delta. \qquad ...(1)$$

Now $\qquad |l| = |l - f(x) + f(x)| \leq |l - f(x)| + |f(x)|. \qquad ...(2)$

From (1) and (2), $|l| < \varepsilon + |f(x)|$, when $0 < |x - a| < \delta$

$\Rightarrow \quad |f(x)| > |l| - \varepsilon$, when $0 < |x - a| < \delta$

$\Rightarrow \quad |f(x)| > |l| - \frac{|l|}{2} = \frac{|l|}{2}$, when $0 < |x - a| < \delta$

Hence $|f(x)| > \frac{1}{2}|l|$, when $0 < |x - a| < \delta$.

Theorem : Prove that $\lim_{x \to c} f(x)$ exists and is equal to a number l if and only if both left limit $\lim_{x \to c-0} f(x)$ and right limit $\lim_{x \to c+0} f(x)$ exist and are equal to l.

OR

Let f be defined on a deleted neighbourhood of c. Show that $\lim_{x \to c} f(x)$ exists and equals l iff $f(c + 0)$, $f(c - 0)$ both exist and are equal to l.

Proof. Condition is necessary

Let $\lim_{x \to c} f(x) = l$. Then for any $\varepsilon > 0$, there exists some $\delta > 0$ such that

$$|f(x) - l| < \varepsilon, \quad \text{when} \quad 0 < |x - c| < \delta$$

or $\qquad |f(x) - l| < \varepsilon$, when $c - \delta < x < c + \delta$, $x \neq c$.

It follows that $|f(x) - l| < \varepsilon$, when $c - \delta < x < c$, $\qquad ...(1)$

and $\qquad |f(x) - l| < \varepsilon$, when $c < x < c + \delta$. $\qquad ...(2)$

From (1) and (2), we see that

$$\lim_{x \to c} f(x) \text{ and } \lim_{x \to c-0} f(x) \text{ both exist and are equal to } l.$$

Condition is sufficient

Let $\lim\limits_{x \to c+0} f(x) = l = \lim\limits_{x \to c+0} f(x)$.

Then for any $\varepsilon > 0$, there exist some $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$|f(x) - l| < \varepsilon, \text{ when } c < x < c + \delta_1, \quad ...(3)$$

and $$|f(x) - l| < \varepsilon, \text{ when } c - \delta_2 < x < c. \quad ...(4)$$

Let $\delta = \min(\delta_1, \delta_2)$. Then $\delta \le \delta_1$ and $\delta_2 \le \delta_2$

$\Rightarrow$ $c + \delta \le c + \delta_1$ and $c - \delta \ge c - \delta_2$ (or $c - \delta_2 \le c - \delta$). ...(5)

From (3), and (5), we obtain

$$|f(x) - l| < \varepsilon, \text{ when } c < x < c + \delta,$$

and $$|f(x) - l| < \varepsilon, \text{ when } c - \delta < x < c.$$

$\therefore$ $$|f(x) - l| < \varepsilon, \text{ when } c - \delta < x < c + \delta,\ x \ne c$$

or $$|f(x) - l| < \varepsilon, \text{ when } 0 < |x - c| < \delta.$$

Hence $\lim\limits_{x \to c} f(x) = l$.

Ex : Show that $\lim\limits_{x \to 0} \dfrac{x - |x|}{x}$ does not exist.

Sol. We have

$$\lim_{x \to 0-} \frac{x - |x|}{x} = \lim_{x \to 0} \frac{x - (-x)}{x} = \lim_{x \to 0} \frac{2x}{x} = 2,$$

and $$\lim_{x \to 0+} \frac{x - |x|}{x} = \lim_{x \to 0} \frac{x - x}{x} = 0.$$

L.H.L $\ne$ R.H.L

Hence $\lim\limits_{x \to 0} \dfrac{x - |x|}{x}$ does not exist.

Infinite Limits or Limits at Infinity

Definition : The expression $\lim\limits_{x \to +\infty} f(x) = l$ means, given any $\varepsilon > 0$, there exists some $M > 0$ such that

$$|f(x) - l| < \varepsilon, \text{ when } x > M.$$

Definition : The expression $\lim\limits_{x \to -\infty} f(x) = l$ means, given any $\varepsilon > 0$, there exists some $M > 0$ such that

$$|f(x) - l| < \varepsilon, \text{ when } x < -M.$$

Definition : The expression $\lim\limits_{x \to a} f(x) = +\infty$ means, given any $M > 0$ (however large), there exists some $\delta > 0$ such that

$$f(x) > m, \text{ whenever } |x - a| < \delta.$$

Definition : The expression $\lim\limits_{x \to a} f(x) = -\infty$ means, given any $M > 0$ (however large), there exists some $\delta > 0$ such that

$$f(x) < -M, \text{ when } |x - a| < \delta.$$

Ex : Show that

(i) $\lim_{x \to 0+} \frac{1}{x} = +\infty$, (ii) $\lim_{x \to 0-} \frac{1}{x} = -\infty$, (iii) $\lim_{x \to 0} \frac{1}{x^2} = +\infty$.

Sol.

(i) Let $M > 0$ be any large number. Let $\delta = 1/M$.

Then $0 < x < \delta \Rightarrow \frac{1}{x} > \frac{1}{\delta} \Rightarrow \frac{1}{x} > M$.

Hence $\lim_{x \to 0+} \frac{1}{x} = +\infty$, by

(ii) Again $-d < x < 0 \Rightarrow -\frac{1}{\delta} > \frac{1}{x} \Rightarrow -M > \frac{1}{x} \Rightarrow \frac{1}{x} < -M$.

Hence $\lim_{x \to 0-} \frac{1}{x} = -\infty$, by

Note : $\lim_{x \to 0} \frac{1}{x}$ does not exist.

(iii) Now $-\delta < x < \delta \Rightarrow \frac{1}{x^2} > \delta \Rightarrow \frac{1}{x^2} > M$

Hence $\lim_{x \to 0} \frac{1}{x^2} = +\infty$, by

Ex : Find $\lim_{x \to 0} \frac{x^2 - 4}{x^2 + 7x + 12}$.

Sol. Dividing numerator and denominator by x^2, we get

$$\lim_{x \to \infty} \frac{x^2 - 4}{x^2 + 7x + 12} = \lim_{x \to \infty} \frac{1 - 4/x^2}{1 + 7/x + 12/x^2}$$

$$= \frac{1 - 4 \lim_{x \to \infty} (1/x^2)}{1 + 7 \lim_{x \to \infty}(1/x) + 12 \lim_{x \to \infty}(1/x^2)}$$

$$= \frac{1 - 0}{1 + 0 + 0} = 1.$$

Ex : Find $\lim_{x \to 2} \frac{9 - x^2}{x + 1}$ and $\lim_{x \to 2} (9 - x^2)(x + 1)$.

Sol. If c is a constant, then $\lim_{x \to x_0} c = c$, and, from Example $\lim_{x \to x_0} x = x_0$. Therefore, from Theorem

$$\lim_{x \to 2} (9 - x^2) = \lim_{x \to 2} 9 - \lim_{x \to 2} x^2$$

$$= \lim_{x \to 2} 9 - \left(\lim_{x \to 2} x\right)^2$$

$$= 9 - 2^2 = 5,$$

and

$$\lim_{x\to 2}(x+1)=\lim_{x\to 2}x+\lim_{x\to 2}1=2+1=3.$$

Therefore,

$$\lim_{x\to 2}\frac{9-x^2}{x+1}=\frac{\lim_{x\to 2}(9-x^2)}{\lim_{x\to 2}(x+1)}=\frac{5}{3}$$

and

$$\lim_{x\to 2}(9-x^2)(x+1)=\lim_{x\to 2}(9-x^2)=\lim_{x\to 2}(x+1)=5\cdot 3=15.$$

Ex : Let

$$g(x)=\frac{2x^2-x+1}{3x^2+2x-1}.$$

it leads to an indeterminate form as x tends to infinity. Rewriting it as

$$g(x)=\frac{2-1/x+1/x^2}{3+2/x-1/x^2}.\quad x\neq 0,$$

we find that

$$\lim_{x\to\infty}g(x)=\frac{\lim_{x\to\infty}2-\lim_{x\to\infty}1/x+\lim_{x\to\infty}1/x^2}{\lim_{x\to\infty}3+\lim_{x\to\infty}2/x-\lim_{x\to\infty}1/x^2}=\frac{2-0+0}{3+0-0}=\frac{2}{3}.$$

4. MONOTONIC FUNCTIONS

A function f is *non-decreasing* on an interval I if

$$f(x_1) \le f(x_2) \text{ whenever } x_1 \text{ and } x_2 \text{ are in I and } x_1 < x_2, \qquad ...(1)$$

Similarly, A function f is non-increasing on I if

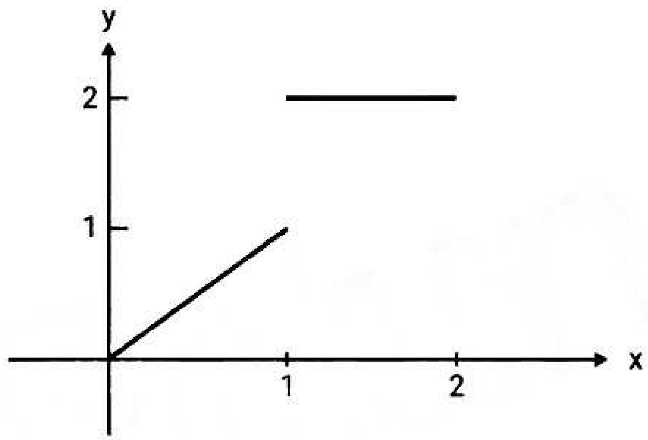
$$f(x_1) \ge f(x_2) \text{ whenever } x_1 \text{ and } x_2 \text{ are in I and } x_1 < x_2. \qquad ...(2)$$

In either case, f is *monotonic* on I. If ≤ is replaced by < in (1), f is said to be *strictly increasing* on I. If ≥ is replaced by > in (2), f is said to be *strictly decreasing* on I. In either of these two cases, f is strictly monotonic on I.

Ex : The function

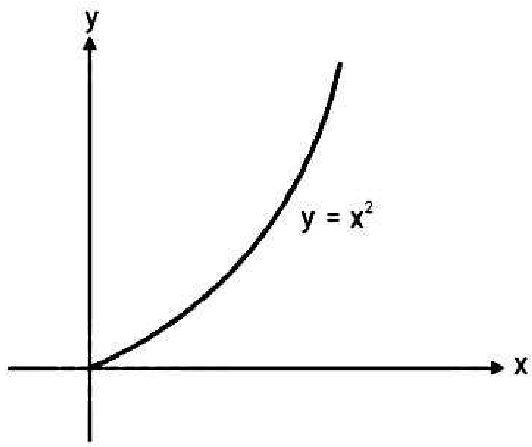
$$f(x) = \begin{cases} x, & 0 \le x < 1, \\ 2, & 1 \le x \le 2, \end{cases}$$

is non-decreasing on I = [0, 2] and – f is non-increasing on I = [0, 2].



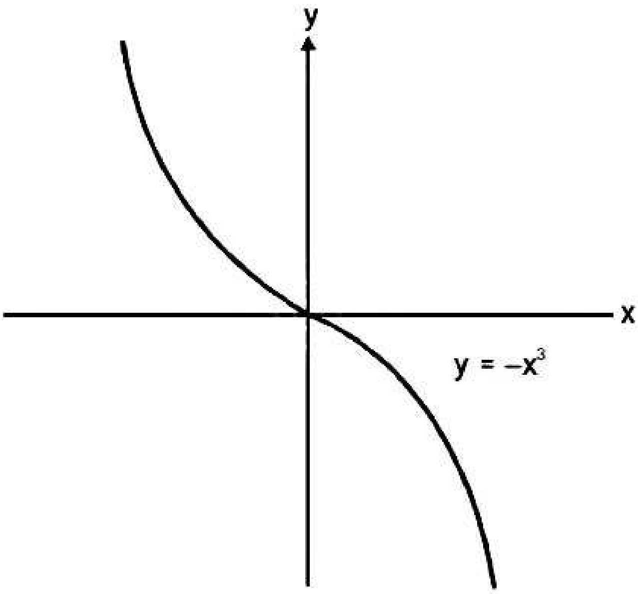
Figure

The function $g(x) = x^2$ is increasing on $[0, \infty)$ Figure,



Figure

and $h(x) = -x^3$ is decreasing on $(-\infty, \infty)$ Figure.



Figure

Properties of Monotonic Functions

(i) If f is monotonic then, for any monotonic sequence $<a_n>$, $<f(a_n)>$ is also monotonic.

(ii) Sum of two monotonically increasing (decreasing) sequence again monotonically increasing (decreasing).

(iii) Difference of two monotonically increasing (decreasing) sequence may not be monotonically increasing (decreasing).

Ex : $f:\left[0,\pi/2\right]\to\mathbb{R}$ & $g:\left[0,\pi/2\right]\to\mathbb{R}$

such that f(x) = sinx & g(x) = x

but (f – g)(x) = sinx – x is not monotonic.

(iv) Similarly product of two monotonically increasing (decreasing) sequences may not be monotonically increasing (decreasing).

Ex : $f:\mathbb{R}\to\mathbb{R}$ & $g:\mathbb{R}\to\mathbb{R}$

such that f(x) = x & g(x) = x

but (fg)(x) = x^2 is not monotonic.

(v) If the function f is increasing (decreasing), then the inverse function $\frac{1}{f}$ is decreasing (increasing).

(vi) Composition of two monotonic functions $f:X\to Y$ & $g:Y\to Z$ is monotonic $gof:X\to Z$.

5. CONTINUOUS FUNCTIONS OF ONE VARIABLE

Consider a function $f : [a, b] \rightarrow \mathbb{R}$, Let $a < c < b$ i.e. $c \in [a,b]$.

Definition : A function f is said to be **continuous** at a point c, if for any $\varepsilon > 0$, there exist some $\delta > 0$ such that

$$| f(x) - f(c) | < \varepsilon, \quad \text{whenever} \quad | x - c | < \delta \qquad ...(1)$$

or $\boxed{\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = f(c).}$

Definition : A function f is said to be **continuous** in an interval [a, b], if it is continuous at every point of the interval [a, b].

Remark : If a function f is not continuous at a point c, then from (1), it follows that there exists some $\varepsilon > 0$ such that for each $\delta > 0$, there is some $y \in [a, b]$ satisfying

$$| f(y) - f(c) | \geq \varepsilon, \text{ when } | y - c | < \delta.$$

Definition : A function f is said to be **discontinuous** at a point x = c, if f is not continuous at x = c.

Remark : The discontinuity of a function f at x = c is obtained in either of the following cases:

1. f is not defined at x = c.
2. $\lim_{x \to c} f(x)$ does not exist.
3. $\lim_{x \to c} f(x)$ exists but $\lim_{x \to c} f(x) \neq f(c)$.

Algebra of Continuous Functions

(i) The identity function f(x) = x is continuous in its domain.

(ii) If f(x) and g(x) are both continuous at x = c, so is f(x) + g(x) at x = c.

(iii) If f(x) and g(x) are both continuous at x = c, so is f(x) * g(x) at x = c.

(iv) If f(x) and g(x) are both continuous at x = c, and $g(x) \neq 0$, then f(x) / g(x) is continuous at x = c.

(v) If f(x) is continuous at x = c, and g(x) is continuous at x = f(c), then the composition g(f(x)) is continuous at x = c.

Ex. The function defined as

$$f(x) = \begin{cases} \dfrac{x^2 - 9}{x - 3}, & x \neq 3 \\ 6 & , \ x = 3 \end{cases},$$

is continuous at x = 3.

Sol. We have

$$\lim_{x \to 3} f(x) = \lim_{x \to 3} \frac{x^2 - 9}{x - 3} = \lim_{x \to 3} (x + 3) = 6.$$

Since $f(3) = 6$, $\quad \lim_{x \to 3} f(x) = f(3)$.

Hence f is continuous at x = 3.

Ex : Show that the function

$$f(x) = \begin{cases} x\sin\dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

is continuous at x = 0.

Sol. We know $\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0}\left(x\sin\dfrac{1}{x}\right) = 0$. {Since sin(1/x) is bounded}

Also $f(0) = 0$. So $\lim\limits_{x\to 0} f(x) = f(0)$.

Hence f is continuous at x = 0.

Ex : Show that

$$f(x) = \begin{cases} \dfrac{|x-a|}{x-a}, & x \neq a \\ 1, & x = a \end{cases}$$

is discontinuous at x = a.

Sol. We know $|x-a| = \begin{cases} (x-a), & \text{if } x \geq a \\ -(x-a), & \text{if } x < a \end{cases}$

Now $\lim\limits_{x\to a-} f(x) = \lim\limits_{x\to a} \dfrac{-(x-a)}{(x-a)} = -1$, as $x \neq a$.

$\lim\limits_{x\to a+} f(x) = \lim\limits_{x\to a} \dfrac{(x-a)}{(x-a)} = 1$, as $x \neq a$.

Thus $\lim\limits_{x\to a} f(x)$ does not exist and so f is discontinuous at x = a.

Ex : Examine the continuity of the function

$$f(x) = \begin{cases} 2x, & \text{if } 0 \leq x \leq 1 \\ 2-x, & \text{if } 1 < x \leq 2 \\ x^2 - 2x, & \text{if } x > 2 \end{cases}$$

at x = 1 and x = 2.

Sol. We have $\lim\limits_{x\to 1-} f(x) = \lim\limits_{x\to 1} 2x = 2$,

$\lim\limits_{x\to 1+} f(x) = \lim\limits_{x\to 1} (2-x) = 1$.

Thus $\lim\limits_{x\to 1} f(x)$ does not exist and so f is discontinuous at x = 1.

Now $\lim\limits_{x\to 2-} f(x) = \lim\limits_{x\to 2} (2-x) = 0$,

$\lim\limits_{x\to 2+} f(x) = \lim\limits_{x\to 2} (x^2 - 2x) = 0$.

Also $f(2) = 2 - 2 = 0.$

Thus $\lim\limits_{x \to 2-} f(x) = \lim\limits_{x \to 2-} f(x) = f(2)$ and so f is continuous at x = 2.

Ex : Show that the function

$$f(x) = \begin{cases} \dfrac{x\, e^{1/x}}{1+e^{1/x}}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

is continuous at x = 0.

Sol. We have

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{x\, e^{1/x}}{1+e^{1/x}} = 0.$$

$$\therefore \quad \lim_{x \to 0} f(x) = f(0). \qquad (\because \ f(0) = 0)$$

Hence f is continuous at x = 0.

Ex : Show that the function

$$f(x) = \begin{cases} \dfrac{e^{1/x}-1}{e^{1/x}+1}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

is discontinuous at x = 0.

Sol. $f(x) = \begin{cases} \dfrac{e^{1/x}-1}{e^{1/x}+1} & x \neq 0 \\ 0 & x = 0 \end{cases}$

By computing two sided limit

Left Hand Limit (LHL)

$$\lim_{x \to 0^-} \frac{e^{1/x}-1}{e^{1/x}+1} \qquad \text{as } x \to 0^- \Rightarrow e^{1/x} \to 0$$

$$= \frac{0-1}{0+1} = -1$$

Right Hand Limit (RHL)

$$\lim_{x \to 0^+} \frac{e^{1/x}-1}{e^{1/x}+1} \qquad \text{as } x \to 0^+ \Rightarrow e^{-1/x} \to 0$$

$$= \lim_{x \to 0^+} \frac{e^{1/x}(1-e^{-1/x})}{e^{1/x}(1+e^{-1/x})}$$

$$= \lim_{x \to 0^+} \frac{(1-e^{-1/x})}{(1+e^{-1/x})} = \frac{1-0}{1+0} = 1$$

RHL $\neq$ LHL $\neq f(0)$

So f is discontinuous at x = 0.

Ex : Show that the function

$$f(x)=\begin{cases}\dfrac{e^{1/x}-e^{-1/x}}{e^{1/x}+e^{-1/x}}, & x\neq 0\\ 1 & , x=0\end{cases}$$

is discontinuous at $x = 0$.

Sol. We know $x \to 0-$ $\Rightarrow$ $e^{1/x} \to 0$ ...(1)

and $x \to 0+$ $\Rightarrow$ $e^{-1/x} \to 0$. ...(2)

$$\therefore \quad \lim_{x\to 0-}\frac{e^{1/x}-e^{-1/x}}{e^{1/x}+e^{-1/x}}=\lim_{x\to 0-}\frac{e^{1-x}.e^{1/x}-1}{e^{1/x}.e^{1/x}+1}=\frac{0.0-1}{0.0+1}=-1, \text{ by (1)}$$

$$\lim_{x\to 0+}\frac{e^{1/x}-e^{-1/x}}{e^{1/x}+e^{-1/x}}=\lim_{x\to 0+}\frac{1-e^{1/x}.e^{-1/x}}{1+e^{-1/x}.e^{-1/x}}=\frac{1-0.0}{1+0.0}=1, \text{ by (2).}$$

Hence $\lim\limits_{x\to 0}\dfrac{e^{1/x}-e^{-1/x}}{e^{1/x}+e^{-1/x}}$ does not exist $(\because -1 \neq 1)$

and so the given function is discontinuous at $x = 0$.

Ex : Examine the continuity of the function :

$$f(x)=\begin{cases}\dfrac{e^{1/x^2}}{1-e^{1/x^2}}, & \text{when } x\neq 0\\ 0 & , \text{when } x=0\end{cases}$$

at $x = 0$.

Sol. Clearly, $x \to 0+ \Rightarrow 1/x^2 \to +\infty \Rightarrow e^{1/x^2} \to +\infty$

$$\Rightarrow 1/e^{1/x^2} \to 0 \Rightarrow e^{-1/x^2} \to 0.$$

$$\lim_{x\to 0+} f(x)=\lim_{x\to 0+}\frac{e^{1/x^2}}{1-e^{1/x^2}}=\lim_{x\to 0+}\frac{1}{e^{-1/x^2}-1}=\frac{1}{0-1}=-1.$$

Similarly, $\lim\limits_{x\to 0-} f(x) = -1$ and so $\lim\limits_{x\to 0} f(x) = -1 \neq f(0)$ $[\because f(0) = 0]$

Hence f is discontinuous at $x = 0$.

Ex : Show that the function f defined on $\mathbb{R}$ by setting

$$f(x) = \frac{x}{1+e^{1/x}}, \text{ when } x \neq 0 \text{ and } f(0) = 0$$

is continuous at $x = 0$.

Sol. $$\lim_{x\to 0-} f(x)=\lim_{x\to 0-}\frac{x}{1+e^{1/x}}=\frac{0}{1+0}=0, \qquad \left(\because \lim_{x\to 0-} e^{1/x}=0\right)$$

$$\lim_{x\to 0+} f(x)=\lim_{x\to 0+}\frac{x}{1+e^{1/x}}=\lim_{x\to 0+}\frac{x\,e^{-1/x}}{e^{-1/x}+1}=\frac{0.0}{0+1}=0. \qquad \left(\because \lim_{x\to 0+} e^{-1/x}=0\right)$$

Also $f(0) = 0$. Hence f is continuous at $x = 0$.

Ex : Show that

$$f(x) = |x| + |x-1|$$

is continuous at $x = 0$ and $x = 1$.

Sol. f(x) = – x – (x – 1) = 1 – 2x, when x < 0; ...(1)

f(x) = x – (x – 1) = 1 when 0 ≤ x < 1 ; ...(2)

Now $\lim_{x \to 0-} f(x) = \lim_{x \to 0} (1) = 1$, by (2).

Also f(0) = | 0 | + | 0 – 1 | = 1.

∴ $\lim_{x \to 0-} f(x) = \lim_{x \to 0+} f(x) = f(0)$.

Hence f is continuous at x = 0.

Now $\lim_{x \to 1-} f(x) = \lim_{x \to 1} (1) = 1$, by (2)

$\lim_{x \to 1+} f(x) = \lim_{x \to 1} (2x - 1) = 1$, by (3)

Also f(1) = | 1 | + | 1 – 1 | = 1.

∴ $\lim_{x \to 1-} f(x) = \lim_{x \to 1+} f(x) = f(1)$.

Hence f is continuous at x = 1.

Ex : Let f be the function defined on $\mathbb{R}$ by setting

f(x) = [x], for all x ∈ $\mathbb{R}$,

where [x] denotes the greatest integer not exceeding x. Show that f is discontinuous at the points x = 0, ± 1, ± 2, ± 3, ... and is continuous at every other point.

Sol. By definition, we have

$$[x] = 0, \text{ for } 0 \le x < 1,$$
$$[x] = 1, \text{ for } 1 \le x < 2,$$
$$[x] = 2, \text{ for } 2 \le x < 3,$$
$$\vdots$$
$$[x] = -1, \text{ for } -1 \le x < 0.$$
$$[x] = -2 \text{ for } -2 \le x - 1 \text{ and so on.}$$

At x = 0

$$\lim_{x \to 0-} f(x) = \lim_{\substack{x \to 0 \\ (x < 0)}} [x] = -1,$$

$$\lim_{x \to 0+} f(x) = \lim_{\substack{x \to 0 \\ (x > 0)}} [x] = 0.$$

Since $\lim_{x \to 0-} f(x) \ne \lim_{x \to 0+} f(x)$, f is discontinuous at 0.

At x = 1

$$\lim_{x \to 1-} f(x) = \lim_{\substack{x \to 1 \\ (x < 1)}} [x] = -2,$$

$$\lim_{x \to 1+} f(x) = \lim_{\substack{x \to -1 \\ (x < -1)}} [x] = -1,$$

So f is discontinuous at –1.

Similarly, f is discontinuous at – 2, – 3, – 4, ...

Let α ∈ $\mathbb{R}$ ~ Z be any real number but not an integer. Then there exists an integer n such that n ≤ α < n + 1. Then

$[x] = n$, for $n \le x < n + 1$.

Now $\lim\limits_{x \to \alpha-} f(x) = \lim\limits_{x \to \alpha+} f(x) = f(\alpha) = n.$

Hence f is continuous at α.

Ex : Let f be the function on [0, 1] defined by

$$f(x) = \begin{cases} (-1)^r & \text{if } \dfrac{1}{r+1} \le x < \dfrac{1}{r}, r = 1, 2, \ldots \\ 0, & \text{if } x = 0 \\ 1 & \text{if } x = 1. \end{cases}$$

Examine the continuity of f at 1, $\frac{1}{2}, \frac{1}{3}, \ldots$

Sol. We observe that

$$f(x) = -1, \text{ if } \frac{1}{2} \le x < 1 \qquad (r = 1)$$

$$= 1, \text{ if } \frac{1}{3} \le x < \frac{1}{2} \qquad (r = 2)$$

$$= -1, \text{ if } \frac{1}{4} \le x < \frac{1}{3} \qquad (r = 3)$$

and so on.

At x = 1.

$$\lim_{x \to 1-} f(x) = \lim_{\substack{x \to 1 \\ (x < 1)}} f(x) = -1. \text{ Also } f(1) = 1.$$

Since $\lim\limits_{x \to 1-} f(x) \ne f(1)$, f is continuous at x = 1.

At x = $\frac{1}{2}$.

$$\lim_{x \to \frac{1}{2}-} f(x) = \lim_{\substack{x \to \frac{1}{2} \\ (x < \frac{1}{2})}} f(x) = 1,$$

$$\lim_{x \to \frac{1}{2}+} f(x) = \lim_{\substack{x \to \frac{1}{2} \\ (x > \frac{1}{2})}} f(x) = -1.$$

Thus f is discontinuous at $\frac{1}{2}$.

Similarly, f is discontinuous at $\frac{1}{3}, \frac{1}{4}, \frac{1}{5} \ldots$

Ex : Show that the function f on [0, 1] defined as

$$f(x) = \frac{1}{2^n}, \text{ when } \frac{1}{2^{n+1}} < x \le \frac{1}{2^n}, (n = 0, 1, 2, \ldots)$$

is discontinuous at $\frac{1}{2}, \left(\frac{1}{2}\right)^2, \left(\frac{1}{2}\right)^3, \ldots$

Sol. We observe that

$$
\begin{aligned}
f(x) &= 1 && \text{when } \frac{1}{2} < x \le 1 && (n = 0)\\
&= \frac{1}{2}, && \text{when } \frac{1}{2^2} < x \le \frac{1}{2} && (n = 1)\\
&= \frac{1}{2^2}, && \text{when } \frac{1}{2^3} < x \le \frac{1}{2^2} && (n = 2) \text{ and so on.}
\end{aligned}
$$

At $x = \frac{1}{2}$.

$$\lim_{x \to \frac{1}{2}-} f(x) = \lim_{x \to \frac{1}{2}} f(x) = \frac{1}{2},$$

$$\lim_{x \to \frac{1}{2}+} f(x) = \lim_{x \to \frac{1}{2}} f(x) = 1.$$

Thus f is discontinuous at $x = \frac{1}{2}$.

At $x = \left(\frac{1}{2}\right)^2$.

$$\lim_{x \to \frac{1}{2^2}-} f(x) = \frac{1}{2^2} \text{ and } \lim_{x \to \frac{1}{2^2}+} f(x) = \frac{1}{2}.$$

Thus f is discontinuous at $x = \left(\frac{1}{2}\right)^2$ and so on.

At $x = \left(\frac{1}{2}\right)^n$.

$$\lim_{x \to \frac{1}{2^n}-} f(x) = \frac{1}{2^n} \text{ and } \lim_{x \to \frac{1}{2^n}+} f(x) = \frac{1}{2^{n-1}}.$$

Hence f is discontinuous at $\left(\frac{1}{2}\right)^n$, n = 1, 2, ...

Ex : Let f be a function defined by :

$$f(x) = \begin{cases} x\sin(1/x) & , \text{ if } x > 0 \\ 0 & , \text{ if } x = 0 \\ 2x\sin(1/x) & , \text{ if } x < 0. \end{cases}$$

Discuss the continuous of f at x = 0.

Sol. We see that

$$\lim_{x \to 0+} f(x) = \lim_{x \to 0} x \sin\frac{1}{x} = 0.$$

$$\lim_{x \to 0-} f(x) = \lim_{x \to 0} 2x \sin\frac{1}{x} = 2 \times 0 = 0. \text{ Also } f(0) = 0.$$

Hence f is continuous at x = 0.

Ex : Examine the continuity of the function :

$$f(x) = 2x - [x] + \sin(1/x),\ x \neq 0\ ;\ f(0) = 0$$

at x = 0 and x = 2, where [x] denotes the greatest integer not greater than x.

Sol. By Example $\lim_{x\to 0} [x]$ and $\lim_{x\to 2} [x]$ do not exist.

Also $\lim_{x\to 0} \sin\frac{1}{x}$ does not exist.

Thus $\lim_{x\to 0} f(x)$ and $\lim_{x\to 2} f(x)$ do not exist and so the given function is discontinuous at x = 0, 2.

Ex : Let f be the function defined on $\mathbb{R}$ by setting

$$f(x) = \begin{cases} x - [x] - \frac{1}{2}, & \text{when } x \text{ is not an integer} \\ 0 & \text{,when } x \text{ is an integer} \end{cases}$$

Show that f is continuous at all points of $\mathbb{R} \sim Z$ and is discontinuous whenever $x \in Z$.

Sol. Let $n \in Z = \{0, \pm 1, \pm 2, \pm 3, ...\}$ be arbitrary.

Then $\lim_{x\to n} [x]$ does not exist and so $\lim_{x\to n} f(x)$ does not exist.

Hence f is discontinuous whenever $n \in Z$.

We observe that

$$[x] = 0, \text{ if } 0 \le x < 1,$$
$$[x] = 1, \text{ if } 1 \le x < 2,$$
$$[x] = 2, \text{ if } 2 \le x < 3,$$
$$\cdots\cdots\cdots\cdots$$
$$[x] = m, \text{ if } m \le x \le m + 1 \text{ and so on.}$$

Using these, we obtain

$$f(x) = x - \frac{1}{2}, \qquad \text{if } 0 \le x < 1,$$

$$f(x) = x - 1 - \frac{1}{2} = x - \frac{3}{2}, \qquad \text{if } 1 \le x < 2,$$

$$f(x) = x - 2 - \frac{1}{2} = x - \frac{5}{2}, \qquad \text{if } 2 \le x < 3 \text{ and so on.}$$

The above relations are all linear and so f is continuous at all positive non-integral points. Similarly, f is continuous at all negative non-integral points, since

$$f(x) = x - (-1) - \frac{1}{2} = x + \frac{1}{2}, \quad \text{if} \quad -1 \le x < 0$$

$$= x - (-2) - \frac{1}{2} = x + \frac{3}{2}, \quad \text{if} \quad -2 \le x < -1 \text{ etc.}$$

Hence f is continuous at all points of $\mathbb{R} \sim Z$.

Types of Discontinuities

(i) The function f is said to have a removable discontinuity at x = c, if $\lim_{x\to c} f(x)$ exists but is not equal to f(c). If we redefine f(c), then f will become continuous.

(ii) f is said to have a discontinuity of the first kind at x = c, if $\lim_{x \to c-} f(x)$ and $\lim_{x \to c+} f(x)$ both exist but are not equal.

(iii) f is said to have a discontinuity of the first kind from the left at x = c, if $\lim_{x \to c-} f(x)$ exists but is not equal to f(c).

(iv) f is said to have a discontinuity of the first kind from the right at x = c, if $\lim_{x \to c+} f(x)$ exists but is not equal to f(c).

(v) f is said to have a discontinuity of the second kind from the left [right] at x = c, if $\lim_{x \to c-} f(x) \left[\lim_{x \to c+} f(x)\right]$ does not exist.

Ex : The function $f(x) = \frac{(\sin 2x)}{x}$, $x \neq 0$ and f(0) = 1 has a removable discontinuity at the origin.

Sol : We have

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{(\sin 2x)}{x} = \lim_{x \to 0} \frac{(\sin 2x)}{2x} 2 = 1.2 = 2.$$

Also f(0) = 1, $\therefore$ $\lim_{x \to 0} f(x) \neq f(0)$.

Hence f has a removable discontinuity at x = 0.

Ex : Examine the continuity of the function f defined by

$$f(x) = \frac{e^{1/x} - e^{-1/x}}{e^{1/x} + e^{-1/x}} \text{ if } x \neq 0,\ f(0) = 0$$

at x = 0. Also discuss the kind of discontinuity if any.

Sol. We have,

$$\lim_{x \to 0-} f(x) = -1 \text{ and so } \lim_{x \to 0-} f(x) \neq f(0);$$

$$\lim_{x \to 0+} f(x) = 1 \text{ and so } \lim_{x \to 0+} f(x) \neq f(0).$$

Hence f has a discontinuity of the first kind from the left and the right at x = 0.

Ex : A function f is defined by

$$f(x) = \begin{cases} -x^2, & \text{if } x \leq 0 \\ 5x - 4, & \text{if } 0 < x \leq 1 \\ 4x^2 - 3x, & \text{if } 1 < x < 2, \\ 3x + 4, & \text{if } x \geq 2 \end{cases}$$

Examine f for continuity at x = 0, 1, 2. Also discuss the kind of discontinuity, if any.

Sol. At x = 0.

$$\lim_{x \to 0-} f(x) = \lim_{x \to 0} (-x^2) = 0,$$

$$\lim_{x \to 0+} f(x) = \lim_{x \to 0} (5x - 4) = -4.$$

Also f(0) = 0.

Thus $\lim\limits_{x \to 0+} f(x) \neq f(0)$ and so f has a discontinuity of the first kind from the right at x = 0.

At x = 1.

$$\lim_{x \to 1-} f(x) = \lim_{x \to 1} (5x - 4) = 1,$$

$$\lim_{x \to 1+} f(x) = \lim_{x \to 1} (4x^2 - 3x) = 1,$$

and $\qquad f(1) = 5 \times 1 - 4 = 1.$

Thus $\lim\limits_{x \to 1-} f = \lim\limits_{x \to 1+} f(x) = f(1)$ and so f is continuous at x = 1.

At x = 2.

$$\lim_{x \to 2-} f(x) = \lim_{x \to 2} (4x^2 - 3x) = 4 \times 4 - 3 \times 2 = 10,$$

$$\lim_{x \to 2+} f(x) = \lim_{x \to 2} (3x - 4) = 10.$$

Also $f(2) = 3 \times 2 + 4 = 10$.

Thus $\lim\limits_{x \to 2-} f(x) = \lim\limits_{x \to 2+} f(x) = f(x)$ and so f is continuous at x = 2.

Ex : Obtain the points of discontinuity of the function f defined on [0, 1] as follows :

$$f(x) = \frac{1}{2} - x, \text{ if } 0 < x < \frac{1}{2}$$

$$= \frac{3}{2} - x, \text{ if } \frac{1}{2} < x < 1$$

$$f(0) = 0, f\left(\frac{1}{2}\right) = \frac{1}{2}, f(1) = 1.$$

Also examine the kind of discontinuities.

Sol. At x = 0.

$$\lim_{x \to 0+} f(x) = \lim_{\substack{x \to 0 \\ (x > 0)}} f(x) = \lim_{x \to 0} \left(\frac{1}{2} - x\right) = \frac{1}{2}.$$

Also $\qquad f(0) = 0. \qquad \therefore \quad \lim\limits_{x \to 0+} f(x) \neq f(0).$

Thus f has a discontinuity of the first kind from the right at 0.

At $x = \frac{1}{2}$.

$$\lim_{x \to \frac{1}{2}-} f(x) = \lim_{x \to \frac{1}{2}} \left(\frac{1}{2} - x\right) = 0,$$

$$\lim_{x \to \frac{1}{2}+} f(x) = \lim_{x \to \frac{1}{2}} \left(\frac{3}{2} - x\right) = 1.$$

$$\therefore \quad \lim_{x \to \frac{1}{2}-} f(x) \neq \lim_{x \to \frac{1}{2}+} f(x).$$

Thus f has a discontinuity of the first kind at x = $\frac{1}{2}$.

At x = 1.

$$\lim_{x \to 1-} f(x) = \lim_{x \to 1}\left(\frac{3}{2} - x\right) = \frac{1}{2}. \text{ Also } f(1) = 1.$$

$$\therefore \qquad \lim_{x \to 1-} f(x) \neq f(1).$$

Thus f has a discontinuity of the first kind from the left at x = 1.

Ex : Examine for continuity at x = 0, the function

$$f(x) = \begin{cases} \dfrac{e^{1/x}}{1 + e^{1/x^2}}, & \text{when } x \neq 0 \\ 0, & \text{when } x = 0. \end{cases}$$

Sol. $$f(x) = \begin{cases} \dfrac{e^{1/x}}{1 + e^{1/x^2}} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

$$\lim_{x \to 0} \frac{e^{1/x}}{1 + e^{1/x^2}}$$

By applying L-Hospital rule–

$$= \lim_{x \to 0} \frac{e^{1/x}\left(-\frac{1}{x^2}\right)}{e^{1/x^2}\left(-\frac{2}{x^3}\right)}$$

$$= \lim_{x \to 0} \frac{xe^{1/x}}{2e^{1/x^2}}$$

$$= 0$$

$\Rightarrow$ f is continuous at x = 0.

Ex : Examine the function

$$f(x) = \frac{e^{1/x}\sin(1/x)}{1 + e^{1/x}}, \ x \neq 0 \ ; f(0) = 0$$

for points of discontinuity, if any.

Sol. We know that

$$x \to 0- \Rightarrow e^{1/x} \to 0 \quad \text{and} \quad x \to 0+ \Rightarrow e^{-1/x} \to 0.$$

$$\therefore \qquad \lim_{x \to 0+} \frac{e^{1/x}\sin(1/x)}{1 + e^{1/x}} = \lim_{x \to 0+} \frac{\sin(1/x)}{e^{-1/x} + 1} = \lim_{x \to 0+} \sin\frac{1}{x},$$

which does not exist. Hence f has a discontinuity of the second kind from the right at x = 0.

Theorems on Continuous Functions

Theorem : (a) If two functions f, g are continuous at a point c, then the functions f + g, f – g, fg are also continuous at c and if g (c) ≠ 0, then f / g is also continuous at c.

(b) Prove that if a function f is continuous at x = a, then | f | is also continuous at x = a. But the converse if not true.

Proof. Since f and g are continuous at c,

$$\lim_{x \to c} f(x) = f(c) \text{ and } \lim_{x \to c} g(x) = g(c). \qquad ...(1)$$

We show that f + g is continuous at c. We have

$$\lim_{x \to c} (f+g)(x) = \lim_{x \to c} [f(x) + g(x)]$$

$$= \lim_{x \to c} f(x) + \lim_{x \to c} g(x)$$

$$= f(c) + g(c), \text{ by (1)}$$

$$\therefore \qquad \lim_{x \to c} (f+g) = (f+g)(c).$$

Thus f + g is continuous at c.

Again, $\lim_{x \to c} (fg)(x) = \lim_{x \to c} f(x)\, g(x)$

$$= \lim_{x \to c} f(x) \,.\, \lim_{x \to c} g(x)$$

$$= f(c)\, g(c) = (fg)(c).$$

Thus fg is continuous at c

Similarly, we can prove the remaining parts.

(b) Since f is continuous at x = a, therefore, for $\varepsilon > 0$ there exists some $\delta > 0$ such that

$$| f(x) - f(a) | < \varepsilon \quad \text{when} \quad 0 < | x - a | < \delta \qquad ...(2)$$

We know $\quad || x | - | y || \le | x - y | \ \forall\ x, y \in \mathbb{R}.$

$\therefore \quad || f(x) | - | f(a) || \le | f(x) - f(a) | < \varepsilon,\ 0 < | x - a | < \delta,$ using (2)

i.e., $\quad || f(x) | - | f(a) || < \varepsilon,$ when $0 < | x - a | < \delta.$

Hence | f | is continuous at x = a.

However, if | f | is continuous at x = a, then f may not be continuous at x = a as seen below :

Let $\quad f(x) = \begin{cases} 1, & \text{when x is rational,} \\ -1, & \text{when x is irrational.} \end{cases}$

Then $| f(x) | = 1\ \forall\ x \in \mathbb{R}$ and so | f | is continuous at every point of $\mathbb{R}$, but f is discontinuous at every point of $\mathbb{R}$.

Remarks : 1. Let f and g be defined on an interval I. If f + g and fg are continuous at a point $p \in I$, then f and g may not be continuous at p as explained by the following examples :

(i) Let $\quad f(x) = \dfrac{e^{1/x} - 1}{e^{1/x} + 1}$, $x \neq 0$ and f(0) = 0 ;

$$g(x) = \frac{1 - e^{1/x}}{e^{1/x} + 1}, \ x \neq 0 \text{ and } g(0) = 0.$$

Thus f and g are discontinuous at x = 0, but f + g = 0 (being a constant function) is continuous at x = 0.

(ii) Let f(x) = g(x) = 1, when x is rational

f(x) = g(x) = –1, when x is irrational.

By Example, f and g are discontinuous at every point of $\mathbb{R}$, but fg = 1 (being a constant function) is continuous at every point of $\mathbb{R}$.

Similarly, we can give examples to show that if f – g, f/g are continuous at a point $p \in I$; then f and g may not be continuous at p.

Notice that in (ii), $f/g = 1\ \forall\ x \in \mathbb{R} \Rightarrow f/g$ continuous $\forall\ x \in \mathbb{R}$.

If in (i), $g(x) = f(x)$, then $f - g = 0 \Rightarrow f - g$ is continuous $\forall\ x \in \mathbb{R}$.

2. A polynomial function $f(x) = a_0 + a_1x + a_2x^2 + ... + a_nx^n$ is continuous for all $x \in \mathbb{R}$.

Clearly, $\lim\limits_{x \to c} x^n = c^n\ \forall\ n \in N \Rightarrow x^n$ is continuous $\forall\ x \in \mathbb{R}$, $\forall\ n \in N$.

Also a_n (being a constant) is continuous. By Theorem , a_nx^n is continuous $\forall\ n \in N$ and $\forall\ x \in \mathbb{R}$. Hence $a_0 + a_1x + a_2x^2 + ... + a_nx^n$ is continuous $\forall\ x \in \mathbb{R}$.

Theorem : A function f defined on an interval I is continuous at a point $p \in I$ if and only if for each sequence $\langle p_n \rangle$ converging to p, the sequence $\langle f(p_n) \rangle$ converges to f(p).

Proof. The condition is necessary.

We are given that the function f is continuous at p and the sequence $\langle p_n \rangle$ converges to p. We shall show that $\langle f(p_n) \rangle$ converges to f(p). Since f is continuous at x = p, therefore for $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$| f(x) - f(p) | < \varepsilon, \quad \text{when } | x - p | < \delta. \qquad ...(1)$$

Since the sequence $\langle p_n \rangle$ converges to p, so there exists some positive integer m such that

$$|p_n - p| < \delta \quad \forall\ n \geq m. \qquad ...(2)$$

Replacing x by p_n in (1), we get

$$| f(p_n) - f(p) | < \varepsilon \text{ when } | p_n - p | < \delta. \qquad ...(3)$$

From (2) and (3), we obtain

$$| f(p_n) - f(p) | < \varepsilon \quad \forall\ n \geq m. \text{ Thus } \langle f(p_n) \rangle - f(p).$$

The conditions is sufficient.

Let $\langle p_n \rangle \to p \quad \Rightarrow \quad \langle f(p_n) \rangle \to f(p)$.

We shall show that f is continuous at x = p.

Let, if possible, f be not continuous at x = p.

Then for some $\varepsilon > 0$ and for each $\delta > 0$, $\exists$ at least one x such that

$$| f(x) - f(p) | \geq \varepsilon, \quad \text{whenever} \quad | x - p | < \delta.$$

Let $\delta = 1/n$. Then $\forall\ n \in N$, there exists $x = p_n$ such that

$$| p_n - p | < 1/n \quad \Rightarrow \quad | f(p_n) - f(p) | \geq \varepsilon$$

i.e., $\lim\limits_{n \to \infty} p_n = p$, but $\lim\limits_{n \to \infty} f(p_n) \neq f(p)$. $(\because\ p_n - p \to 0 \text{ as } n \to \infty)$

Thus we have proved that there is a sequence $\langle p_n \rangle$ such that $\langle p_n \rangle$ converges to p, but the sequence $\langle f(p_n) \rangle$ does not converge to f(p). This is a contradiction to the given condition. Hence f is continuous at x = p.

Theorem : Let f be a function defined on an interval I and $p \in I$. Let g be a function defined on an interval J such that $f(I) \subseteq J$. If f is continuous at p and g is continuous at f(p), then show that gof is continuous at p.

Proof. Let $\langle p_n \rangle$ be any sequence in I such that $p_n \to p$. ...(1)

Since f is continuous at p, so by Theorem, $f(p_n) \to f(p)$.

Since $f(I) \subseteq J$, so $\langle f(p_n) \rangle$ is a sequence in J converging to $f(p) \in J$.

Since g is continuous at f(p) and $f(p_n) \to f(p)$ in J, so

$$g(f(p_n)) \to g(f(p))$$

or $$(gof)(p_n) \to (gof)(p). \quad ...(2)$$

[By definition, $(gof)(x) = g(f(x))\ \forall\ x \in I$.]

From (1) and (2), it follows that gof is continuous at p

Ex : F be a function on $\mathbb{R}$ defined by

$$f(x) = \begin{cases} 1, & \text{when } x \text{ is rational} \\ -1, & \text{when } x \text{ is irrational.} \end{cases}$$

f is discontinuous at every point of $\mathbb{R}$

Hint. Since **Q, R-Q** are dense in **R** , therefore for every **a** ε **R** we can always find sequences from **Q & R-Q** that converge to **a.**

Ex : Show that the function f defined by

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$$

is discontinuous at every point.

Sol. Case I. Let a be any rational number so that $f(a) = 1$.

Then any nbd. $(a - 1/n, a + 1/n)$ of a contains an *irrational number* a_n for each $n \in N$ i.e.,

$$a_n \in (a - 1/n, a + 1/n) \Rightarrow |a_n - a| < \frac{1}{n}$$

$\Rightarrow \quad |a_n - a| \to 0$ as $n \to \infty\ \langle a_n \rangle \to a$.

Now $f(a_n) = 0\ \forall\ n$ $(\because\ a_n$ is irrational) and $f(a) = 1$

$\Rightarrow \quad \lim\limits_{n \to \infty} f(a_n) = 0 \neq f(a) \Rightarrow \langle f(a_n) \rangle$ does not converge to $f(a)$.

Thus f is discontinuous at all rational points.

Case II. Let b be any irrational number so that $f(b) = 0$.

As argued earlier, we can choose a *rational number* b_n such that

$$|b_n - b| < \frac{1}{n}\ \forall\ n \in N$$

$\Rightarrow \quad |b_n - b| \to 0$ as $n \to \infty \Rightarrow \langle b_n \rangle \to b$.

Now $f(b_n) = 1\ \forall\ n$ $(\because\ b_n$ is rational) and $f(b) = 0$

$\Rightarrow \quad \lim\limits_{n \to \infty} f(b_n) = 1 \neq f(b) \Rightarrow \langle f(b_n) \rangle$ does not converge to $f(b)$.

Thus f is continuous at all irrational points.

Hence the given function is discontinuous at every point of $\mathbb{R}$.

Ex : Show that the function f defined on $\mathbb{R}$ by

$$f(x) = \begin{cases} x, & \text{when } x \text{ is irrational} \\ -x, & \text{when } x \text{ is rational} \end{cases}$$

is continuous only at $x = 0$.

Sol. Case I : Let $a \neq 0$ be any rational number so that

$$f(a) = -a.$$

Then any nbd. $(a - 1/n, a + 1/n)$ of a contains an *irrational number* a_n for each $n \in N$ i.e.,

$$a_n \in (a - 1/n,\ a + 1/n) \quad \Rightarrow \quad |a_n - a| < \frac{1}{n}$$

$\Rightarrow$ $|a_n - a| \to 0$ as $n \to \infty$ $\Rightarrow$ $\langle a_n \rangle \to a$.

Now $f(a_n) = a_n \ \forall\ n$ $(\because\ a_n$ irrational)

$\Rightarrow$ $\lim\limits_{n\to\infty} f(a_n) = \lim\limits_{n\to\infty} a_n = a \neq f(a)$ $(\because\ f(a) = -a)$

$\Rightarrow$ $\langle f(a_n) \rangle$ does not converge to f(a), when $\langle a_n \rangle \to a$.

So f is discontinuous at all non-zero rational points.

Case II. Let $b \neq 0$ be any irrational number so that $f(b) = b$.

As argued earlier, there exists a *rational number* $b_n \ \forall\ n \in N$ such that

$$|b_n - b| < \frac{1}{n} \ \Rightarrow\ |b_n - b| \to 0 \text{ as } n \to \infty \ \Rightarrow\ \langle b_n \rangle \to b.$$

Now $f(b_n) = b_n \ \forall\ n$ $(\because\ b_n$ is rational)

$\Rightarrow$ $\lim\limits_{n\to\infty} f(b_n) = \lim\limits_{n\to\infty} b_n = b \neq 0$

$\Rightarrow$ $\langle f(b_n) \rangle$ does not converge to f(b). $(\because\ f(b) = 0)$

Thus f is discontinuous at every non-zero irrational point.

Now we shall prove that f is continuous at x = 0.

We have $f(0) = 0$ and

$|f(x) - f(0)| = |f(x)| = |x|$, if x is rational

$= 0$, if x is irrational

Let $\varepsilon > 0$. Then $|f(x) - f(0)| < \varepsilon$, for $|x - 0| < \delta$ $(\delta = \varepsilon)$.

Hence f is continuous only at x = 0.

NOTE :we can also use sequential approach to deal with such questions on which f(x) is defined explicitly on all rationals & irrationals. It will consume time in exam.

As let **a** ε **R**

Then there exist $\boldsymbol{<a_n>}$ ε **Q** such that $\boldsymbol{<a_n> \to a}$

Similarly, there exist $\boldsymbol{<b_n>}$ ε **R-Q** such that $\boldsymbol{<b_n> \to a}$

now to find points of continuity we first look for the points where limit exist

for that **lim** $\boldsymbol{<f(a_n)>}$ **= lim** $\boldsymbol{<f(b_n)>}$ **=> a = -a => a=0.**

It is the only point where continuity is to be checked.

since **0** ε **Q => f(0)=0 = lim** $\boldsymbol{<f(a_n)>}$**=lim** $\boldsymbol{<f(b_n)>}$ $\forall$ $\boldsymbol{<a_n>}$ ε **Q** such that $\boldsymbol{<a_n> \to 0}$ $\forall$ $\boldsymbol{<b_n>}$ ε **R-Q** such that $\boldsymbol{<b_n> \to 0}$.

Hence f is continuous only at x = 0.

Ex : Let f be a function defined on (0, 1) by :

$$f(x) = \begin{cases} 0, & \text{if } x \text{ is irrational} \\ 1/q, & \text{if } x = p/q, \text{ where p and q are positive integers having no common factor.} \end{cases}$$

Prove that f is continuous at each irrational point and discontinuous at each rational point.

Sol. Case I. Let $a \in (0, 1)$ be any rational number so that $a = p/q$, where p and q are positive integers having no common factor. Then $f(a) = 1/q$ (as given).

Any nbd. $(a - 1/n,\ a + 1/n)$ of a contains an *irrational number* a_n, for each $n \in N$ i.e.,

$$a_n \in (a - 1/n,\ a + 1/n) \quad \Rightarrow \quad |a_n - a| < \frac{1}{n}$$

$\Rightarrow \quad |a_n - a| \to 0$ as $n \to \infty \quad \Rightarrow \quad \langle a_n \rangle \to a$.

Now $\quad f(a_n) = 0 \quad \forall \; n \qquad\qquad (\because \; a_n$ is irrational)

$\Rightarrow \quad \langle f(a_n) \rangle \to 0 \neq f(a) \qquad\qquad (\because \; f(a) = 1/q > 0.]$

Hence f is discontinuous at every rational point in (0, 1).

Case II. Let $b \in (0, 1)$ be any irrational number, so that

$$f(b) = 0$$

Let $\varepsilon > 0$ be given. We can choose a positive integer n such that

$$1/n < \varepsilon.$$

It is clear that there an can be only a finite number of rational numbers p/q in (0, 1) such that $q < n$.

We can, therefore, find some $\delta > 0$ such that no rational number l/m in $(b - \delta, b + \delta)$ has its denominator less than n i.e., $m \geq n$.

Thus $\quad | x - b | < \delta \quad \Rightarrow \quad | f(x) - f(b) | = | f(x) - 0 | = 0 < \varepsilon, \qquad$ ...(1)

if x is irrational.

If $x = l/m$ is rational such that $| x - b | < \delta$, then

$| x - b | < \delta \quad \Rightarrow \quad | f(x) - f(b) = | f(x) | < 1/m \leq 1/n < \varepsilon. \qquad$ ...(2)

$(\because \; m \geq n)$

From (1) and (2), we have

$$| x - b | < \delta \quad \Rightarrow \quad | f(x) - f(b) | < \varepsilon.$$

Hence f is continuous at every irrational point b in (0, 1).

Ex : Show that the function f defined as

$$f(x) = \begin{cases} x, & \text{if } x \text{ is rational} \\ 1 - x, & \text{if } x \text{ is irrational} \end{cases}$$

is continuous only at $x = \frac{1}{2}$.

Sol. Case I. Let $a \neq \frac{1}{2}$ be any rational number. Then $f(a) = a$.

We can find an *irrational number* a_n for each $n \in N$ such that

$$a_n \in (a - 1/n, a + 1/n) \Rightarrow | a_n - a | < \frac{1}{n} \Rightarrow \langle a_n \rangle \to a.$$

Now $\quad f(a_n) = 1 - a_n \; \forall \; n \qquad\qquad (\because \; a_n$ is irrational)

$\Rightarrow \lim\limits_{n \to \infty} f(a_n) = 1 - \lim\limits_{n \to \infty} a_n = 1 - a \neq a.$

(Observe that if $a = \frac{1}{2}$, then $1 - a = a$).

$\Rightarrow \quad \langle f(a_n) \rangle$ does not converge to f(a). $\qquad (\because \; f(a) = a)$

Thus f is discontinuous at all rational points $a \neq \frac{1}{2}$.

Now we shall show that f is continuous at $x = \frac{1}{2}$ only.

We have $|f(x) - f\left(\frac{1}{2}\right)| = |x - \frac{1}{2}|$, if x is rational $\left(\because f\left(\frac{1}{2}\right) = \frac{1}{2}\right)$

$$= \left|1 - x - \frac{1}{2}\right| = \left|\frac{1}{2} - x\right| = \left|x - \frac{1}{2}\right|, \text{ if x is irrational.}$$

Thus $|f(x) - f\left(\frac{1}{2}\right)| = |x - \frac{1}{2}|$.

Let $\varepsilon > 0$ be given. Choose $\delta = \varepsilon > 0$. Then

$$|f(x) - f\left(\frac{1}{2}\right)| < \varepsilon, \quad \text{when} \quad |x - \frac{1}{2}| < \delta.$$

Hence f is continuous only at $\frac{1}{2}$.

Ex : Let f satisfy

$$f(x + y) = f(x) + f(y) \quad \forall \; x, y \in \mathbb{R}. \qquad ...(1)$$

Show that if f is continuous at a point c, then f is continuous at all points of $\mathbb{R}$.

Sol. (i) Taking $x = y = 0$ in (1), we obtain

$$f(0) = f(0) + f(0) \quad \Rightarrow \quad f(0) = 0. \qquad ...(2)$$

Taking $y = -x$ in (1), we obtain

$$f(0) = f(x) + f(-x) \quad \Rightarrow \quad 0 = f(x) + f(-x), \text{ using (2)}$$

$\therefore$ $f(-x) = -f(x) \; \forall \; x \in \mathbb{R}$. ...(3)

Let $\langle c_n \rangle$ be any sequence of real numbers such that $c_n \to 0$.

Then $c_n + c \to 0 + c = c$

$\Rightarrow$ $f(c_n + c) \to f(c)$, as f is continuous at c

$\Rightarrow$ $f(c_n) + f(c) \to f(c)$, using (1)

$\Rightarrow$ $f(c_n) \to 0$.

Thus $c_n \to 0 \Rightarrow f(c_n) \to 0$. ...(4)

Let x be any real number. We shall show that f is continuous at x. Let $\langle x_n \rangle$ be a sequence of real umbers such that $x_n \to x$

$\Rightarrow$ $x_n - x \to 0$

$\Rightarrow$ $f(x_n - x) \to 0$, using (4)

$\Rightarrow$ $f(x_n) + (-x) \to 0$, using (1)

$\Rightarrow$ $f(x_n) - f(x) \to 0$, using (3)

$\Rightarrow$ $f(x_n) \to f(x)$.

Since $x_n \to x \Rightarrow f(x_n) \to f(x)$, f is continuous for all $x \in \mathbb{R}$.

Ex : Prove that the function $h(x) = \sqrt{\sin x}$ is continuous on $[0, \pi]$.

Sol. First of all, we show that $f(x) = \sqrt{x}$ is continuous $\forall \; x \geq 0$.

Let $c > 0$ so that $f(c) = \sqrt{c}$. For $x \geq 0$, we have

$$|f(x) - f(c)| = |\sqrt{x} - \sqrt{c}| = \left|\frac{x - c}{\sqrt{x} + \sqrt{c}}\right| < \frac{1}{\sqrt{c}} |x - c|.$$

Let $\varepsilon > 0$ be given. Then

$|f(c) - f(c)| < \varepsilon$, whenever $|x - c| < \delta$, $\delta = \sqrt{c}\ \varepsilon > 0$.

Hence f is continuous at $c > 0$.

Obviously, $\lim_{x \to 0} f(x) = 0 = f(0)$. Hence f is continuous for all $x \geq 0$.

Next we show that $g(x) = \sin x$ is continuous for all x Î $\mathbb{R}$.

For $c \in \mathbb{R}$, we have

$$|g(x) - g(c)| = |\sin x - \sin c| = 2\left|\cos\frac{x+c}{2}\sin\frac{x-c}{2}\right|$$

$$\leq 2\left|\sin\frac{x-c}{2}\right|, \text{ since } |\cos x| \leq 1\ \forall\ x \in \mathbb{R}$$

$$\leq 2\left|\frac{x-c}{2}\right|, \text{ since } |\sin x| \leq |x|$$

$$= |x - c|,$$

Let $\varepsilon > 0$ be given. Then $|g(x) - g(c)| < \varepsilon$ for $|x - c| < \delta$, $\delta = \varepsilon > 0$.

Hence g is continuous for all $c \in \mathbb{R}$.

It is clear that h = fog, where $g(x) = \sin x$, $x \in [0, \pi]$ and $f(x) = \sqrt{x}$, $x \geq 0$. By Theorem $h(x) = \sqrt{\sin x}$ is continuous on $[0, \pi]$.

Ex : Given examples of two discontinuous functions f and g such that

(i) fog is continuous but gof is not continuous.

(ii) fog and gof are both continuous.

Sol.

(i) We define two functions f and g as follows :

$$f(x) = \begin{cases} 0, & \text{when } x \in Q \\ x, & \text{when } x \in R \sim Q \end{cases}$$

and
$$g(x) = \begin{cases} 1, & \text{when } x \in Q \\ 0, & \text{when } x \in R \sim Q \end{cases}$$

Then f is discontinuous at each non-zero point of $\mathbb{R}$ and g is discontinuous at each of $\mathbb{R}$

If $x \in Q$, then $(fog)(x) = f(g(x)) = f(1) = 0$.

If $x \in \mathbb{R} \sim Q$, then $(fog)(x) = f(g(x)) = f(0) = 0$.

$\therefore$ $(fog)(x) = 0\ \forall\ x \in \mathbb{R}$. Hence fog is continuous $\forall\ x \in \mathbb{R}$.

If $x \in Q$, then $(gof)(x) = g(f(x)) = g(0) = 1$.

If $x \in \mathbb{R} \sim Q$, then $(gof)(x) = g(f(x)) = g(0) = 0$.

Thus
$$(gof)(x) = \begin{cases} 1, & \text{when } x \in Q \\ 0, & \text{when } x \in R \sim Q. \end{cases}$$

By Example gof is discontinuous $\forall\ x \in \mathbb{R}$.

(ii) We define two functions f and g on $\mathbb{R}$ as follow :

$$f(x) = \begin{cases} 1, & \text{when } x \in Q \\ -1, & \text{when } x \in R \sim Q \end{cases}$$

and $g(x) = \begin{cases} -1, & \text{when } x \in Q \\ 1, & \text{when } x \in R \sim Q. \end{cases}$

Thus f and g are discontinuous at all points of $\mathbb{R}$.

If $x \in Q$, then $(fog)(x) = f(g(x)) = f(-1) = 1$. $(\because -1 \in Q)$

If $x \in \mathbb{R} \sim Q$ then $(fog)(x) = f(g(x)) = f(1) = 1$. $(\because 1 \in Q)$

$\Rightarrow (fog)(x) = 1 \ \forall x \in \mathbb{R} \Rightarrow$ fog is a constant function on $\mathbb{R}$.

It may be noticed that $(gof)(x) = -1 \ \forall x \in \mathbb{R}$ and so gof is continuous at all points of $\mathbb{R}$.

Theorem : If a function f is continuous on a closed bounded interval [a, b], then it is bounded in [a, b].

Proof. Let, if possible, f be *not* bounded above in [a, b].

Then for *each* positive integer n, we can find a point $x_n \in [a, b]$, such that

that $f(x_n) > n \ \forall \ n \in N$. ...(1)

Since $x_n \in [a, b]$ for each $n \in N$, $a \leq x_n \leq b \ \forall n \in N$.

Thus $\langle x_n \rangle$ is a bounded sequence and so it must have a limit point say p (Bolzano-Weierstrass Theorem).

Obviously, p is a limit point of [a, b]. [Result (e)]

Now [a, b] being a closed interval is a closed set and so $p \in [a, b]$.

Since p is a limit point of the sequence $\langle x_n \rangle$, therefore, there exists a subsequence $\langle x_{n_k} \rangle$ of áx$_n$ñ such that $x_{n_k} \to p$. ...(2)

From (1), $f(x_{n_k}) > n_k$ for all k $\Rightarrow$ $< f(x_{n_k}) >$ diverges to ∞

$\Rightarrow$ $f(x_{n_k})$ does not converge to f(p). ...(3)

From (2) and (3), it follows that f is not continuous at the point $p \in [a, b]$

This is contrary to the hypothesis that f is continuous at every point of [a, b]. Hence f is bounded above on [a, b].

Now we show that f is bounded below on [a, b].

Since f is continuous in [a, b], – f is also continuous in [a, b] and so – f is bounded above . Consequently, there exists some $k \in \mathbb{R}$ such that

$-f(x) \leq k \ \forall x \in [a, b]$

$\Rightarrow$ $f(x) \geq -k$ or $-k \leq f(x) \ \forall x \in [a, b]$

$\Rightarrow$ f is bounded below in [a, b].

Hence f is bounded in [a, b].

Theorem : If a function f is continuous an a closed bounded interval [a, b], then it attains its bounds in [a, b].

Proof. Since f is continuous in [a, b], therefore, it is bounded in [a, b]. Let M = sup f and m = inf f.

$\therefore$ $m \leq f(x) \leq M \ \forall x \in [a, b]$.

We shall show that there exist points $\alpha, \beta \in [a, b]$ such that

$$f(\alpha) = M \quad \text{and} \quad f(\beta) = m.$$

We shall prove it by contradiction.

Let $f(x) \neq M \ \forall \ x \in [a, b]$

$\Rightarrow$ $M - f(x) \neq 0, \ \forall \ x \in [a, b]$.

Since f is given to be continuous in [a, b] and M, being a constant function, is also continuous in [a, b], therefore M – f(x) is continuous in [a, b]. Since $M - f(x) \neq 0,\ \forall\ x \in [a, b]$, therefore

$\frac{1}{M-f(x)}$ is continuous in [a, b].

$\Rightarrow$ $\frac{1}{M-f(x)}$ is bounded in [a, b]

Let $K = \sup \frac{1}{M-f(x)}$ and $k = \inf \frac{1}{M-f(x)}$.

$\Rightarrow$ $\frac{1}{M-f(x)} \leq K\ \ \forall\ \ x \in [a, b] \ \Rightarrow\ M - f(x) \geq \frac{1}{K}\ \forall\ x \in [a, b]$

$\Rightarrow$ $f(x) \leq M - \frac{1}{K}\ \ \forall\ \ x \in [a, b]$.

$\Rightarrow$ $M - \frac{1}{K}$ (< M) is an upper bound of f, which contradicts the fact that M is the l.u.b. of f.

Hence there exists some $\alpha \in [a, b]$ such that $f(\alpha) = M$.

Similarly, there exists some $\beta \in [a, b]$ such that $f(\beta) = m$.

Remark : If a function is not continuous on a closed interval, then it may not attain its bounds as shown below :

1. The function $f(x) = x\ \forall\ x \in [0, 1)$ is continuous and bounded in [0, 1), $0 \leq f(x) < 1\ \forall\ x \in [0, 1)$; inf f = 0 and sup f = 1. Here inf f is attained but sup f is not attained, since there is no point of [0, 1) at which f(x) = 1.

   (Observe that the domain of f is not a closed interval)
2. The function $f(x) = x\ \forall\ x \in (0, 1]$ is continuous and bounded in (0, 1], attains the supremum 1 and does not attain the infimum 0.
3. The function $f(x) = x\ \forall\ x \in (0, 1)$ is continuous and bounded in (0, 1), but attains neither the supremum 1 nor the infimum 0.
4. The function $f(x) = 1/x\ \ \forall\ x \in (0, 1]$. Here f attains the supremum 1 and does not attain the infimum 0.

Ex. Prove that if f is continuous in [a, b] and c is the infimum of f in [a, b], then there exists an x_0 in [a, b] such that $f(x_0) = c$.

Proof. We have c = int f in [a, b] i.e., $c \leq f(x)\ \forall\ x \in [a, b]$.

Let, if possible, $f(x) \neq c\ \forall\ x \in [a, b]$. It follows that

$\frac{1}{f(x)-c}$ is continuous in [a, b]

$\Rightarrow$ $\frac{1}{f(x)-c}$ is bounded above in [a, b]

$\Rightarrow$ $\frac{1}{f(x)-c} \leq k\ \forall\ \ x \in [a, b]$, k being some real number

$\Rightarrow$ $f(x) - c \geq \frac{1}{k}\forall\ x \in [a, b] \Rightarrow c + \frac{1}{k} \leq f(x)\ \forall\ x \in [a, b]$

$\Rightarrow$ $c + (1/k)$ is a lower bound of f in [a, b], where $c + (1/k) > c$.

The contradicts the given condition $c = \inf f$. Hence there exists some $x_0 \in [a, b]$ such that $f(x_0) = c$.

Theorem : If a function f is continuous in [a, b] and $c \in (a, b)$ such that $f(c) \neq 0$, then there exists some $\delta > 0$ such that f(x) has the same sign as f(c) for all $x \in (c - \delta, c + \delta)$.

Proof. Since f is continuous at c, for any $\varepsilon > 0$, $\exists$ some $\delta > 0$, such that

$$|f(x) - f(c)| < \varepsilon, \text{ when } \quad |x - c| < \delta.$$

i.e., $$f(c) - \varepsilon < f(x) < f(c) + \varepsilon, \text{ when } \quad c - \delta < x < c + \delta. \quad ...(1)$$

Case I. Let $f(c) > 0$. We choose $\varepsilon > 0$ such that

$$\varepsilon < f(c) \quad \Rightarrow \quad f(c) - \varepsilon > 0. \quad ...(2)$$

From (1) and (2), $f(x) > f(c) - \varepsilon > 0$, when $x \in (c - \delta, c + \delta)$.

Case II. Let $f(c) < 0 \Rightarrow -f(c) > 0$. We choose $\varepsilon > 0$ such that

$$\varepsilon < -f(c) \quad \Rightarrow \quad f(c) + \varepsilon < 0. \quad ...(3)$$

From (1) and (3), $f(x) < f(c) + \varepsilon < 0$, when $x \in (c - \delta, c + \delta)$.

Hence f(x) has the same sign as f(c) $\forall x \in (c - \delta, c + \delta)$.

Corollary. (i) If a function f is continuous at $x = a$ and $f(a) \neq 0$, then there exists some $\delta > 0$, such that f(x) has the same sign as f(a) $\forall x \in [a, a + \delta)$.

(ii) If a function f is continuous at $x = b$ and $f(b) \neq 0$, then there exists some $\delta > 0$, such that f(x) has the same sign as f(b) $\forall x \in (b - \delta, b]$.

Theorem : If a function f is continuous in [a, b] and f(a) and f(b) are of opposite signs, then there exists some point $c \in (a, b)$ such that

$$f(c) = 0.$$

Proof. Since f(a) and f(b) are of opposite signs, we may take

$$f(a) > 0 \quad \text{and} \quad f(b) < 0.$$

Let S be a subset of [a, b] defined as follows :

$$S = \{x : a \leq x \leq b \text{ and } f(x) > 0\}. \quad ...(1)$$

Then S is non-empty $[\because f(a) > 0 \Rightarrow a \in S]$

and S is bounded above with b as its upper bound.

By order-completeness property of $\mathbb{R}$, S has the supremum.

Let $c = \sup S$, $c \in [a, b]$.

Now we shall prove that $c \neq a$ and $c \neq b$, so that $c \in (a, b)$.

Since f is continuous at $x = a$ and $f(a) > 0$, $\exists$ a $\delta_1 > 0$ such that

$$f(x) < 0 \quad \forall \quad x \in (b - \delta_2, b], \text{ [by corollary of Theorem]} \quad ...(2)$$

$\Rightarrow$ $b - \delta_2$ is an upper bound of S, for otherwise, there exists some $y \in S$ such that $y > b - \delta_2$.

i.e., $$b - \delta_2 < y \leq b \Rightarrow f(y) < 0, \text{ by (2)}. \quad ...(3)$$

As $y \in S$, $f(y) > 0$, by (1). This contradicts (3).

Since $c = \sup S$, so $c \leq b - \delta_2 \Rightarrow c \neq b$.

Thus $c \neq a$ and $c \neq b \Rightarrow c \in (a, b)$.

Finally, we show that $f(c) = 0$.

Case I. Let $f(c) > 0$

Since f is continuous at c and $f(c) > 0$, so $\exists$ a $\delta_3 > 0$ such that

$$f(x) > 0 \quad \forall \quad x \in (c - \delta_3, c + \delta_3). \quad ...(4)$$

Let α be a number such that $c < \alpha < c + \delta_3$. ...(5)

$\therefore \quad \alpha \in (c, c + \delta_3) \Rightarrow f(\alpha) > 0$, by (4) $\Rightarrow \alpha \in S$, by (1).

$\therefore \quad \alpha \le c \ (\because \ c = \sup S)$ or $c \ge \alpha$, which contradicts (5).

Thus $f(c) \ge 0$.

Case II. Let $f(c) < 0$.

Since f is continuous at $x = c$ and $f(c) < 0$, so $\exists$ a $\delta_4 > 0$ such that

$$f(x) < 0 \ \forall \ x \in (c - \delta_4, c + \delta_4). \qquad ...(6)$$

Now $c = \sup S \Rightarrow \exists$ at least one $\beta \in S$ such that $c - \delta_4 < \beta \le c$

$\Rightarrow \quad \beta \in (c - \delta_4, c] \Rightarrow f(\beta) < 0$, by (6).

But $\quad \beta \in S \Rightarrow f(\beta) > 0$, which

$\therefore \quad f(c) \le 0$. Also $f(c) \ge 0$.

Hence $f(c) = 0$ for some $c \in (a, b)$.

Theorem by (Inverse Function Theorem)

If a function f defined on the closed interval [a, b] is continuous on [a, b] and one-to-one, the f^{-1} is also continuous.

Ex : Let f be continuous on [0, 1] and let f(x) be in [0, 1] for each x in [0, 1] then f(x) = x for some x in [0, 1]

Sol : We are given that $f(x) \in [0, 1] \ \forall \ x \in [0, 1]$

i.e., $\quad 0 \le f(x) \le 1 \quad \forall \quad [0, 1]. \qquad ...(1)$

If $f(0) = 0$ or $f(1) = 1$, then the problem is solved. Otherwise, we have

$$f(0) > 0 \text{ and } f(1) < 1, \text{ by (1).} \qquad ...(2)$$

Let $\quad g(x) = f(x) - x \quad \forall \quad x \in [0, 1]$

This g is continuous on [0, 1] and

$$g(0) = f(0) - 0 > 0, \ g(1) = f(1) - 1 < 0.$$

there exists some $c \in (0, 1)$ such that

$$g(c) = 0 \Rightarrow f(c) - c = 0.$$

Hence $\quad f(c) = c \quad$ for some $\quad c \in (0, 1)$.

Ex : Show that if f and g are continuous on [a, b] and if $f(a) < g(a)$ and $f(b) > g(b)$, then there exists some $c \in (a, b)$ satisfying $f(c) = g(c)$.

Sol. Let $\quad h(x) = f(x) - g(x) \quad \forall \quad x \in [a, b]. \qquad ...(1)$

Since f and g are continuous on [a, b], so by (1), h is continuous on [a, b].

Also $\quad h(a) = f(a) - g(a) < 0, \qquad (\because \ f(a) < g(a))$

and $\quad h(b) = f(b) - g(b) > 0. \qquad (\because \ f(b) > g(b))$

Thus h is continuous on [a, b] where h(a), h(b) are of opposite signs.

Hence, by Theorem there exists some $c \in (a, b)$ such that

$$h(c) = 0 \Rightarrow f(c) - g(c) = 0 \Rightarrow f(c) = g(c).$$

Ex : Let f be a continuous function on [– 1, 1] such that $\{f(x)\}^2 + x^2 + 1$ for all x in [–1, 1]. Show that either $f(x) = \sqrt{1-x^2}$ for all x in [– 1, 1] or $f(x) = -\sqrt{1-x^2}$ for all x in [– 1, 1].

Sol. Let, if possible, there exists two points x_1, x_2 in [– 1, 1] such that

$$f(x_1) = \sqrt{1-x_1^2} \text{ and } f(x_2) = -\sqrt{1-x_2^2}.$$

Then $\quad f(x_1)\, f(x_2) < 0.$

Since f is continuous in [– 1, 1] so f is continuous in $[x_1, x_2]$ and $f(x_1)\, f(x_2) < 0$. By Theorem $\exists$ some $c \in (x_1, x_2)$ such that $f(c) = 0$.

We are given $\{f(x)\}^2 + x^2 = 1 \quad \forall \; x \in [-1, 1]$.

In particular, $\{(f(x)\}^2 + c^2 = 1 \qquad (\because \; c \in (-1, 1))$

$\Rightarrow \; c^2 = 1 \qquad (\because \; f(c) = 0)$

$\Rightarrow \; c = \pm 1$, which is impossible.

Ex. Discuss the nature of discontinuity of the function f defined by

$$f(x) = \lim_{n \to \infty} \frac{\log(2 + x) - x^{2n} \sin x}{1 + x^{2n}}, 0 \le x \le \frac{\pi}{2} \text{ at } x = 1.$$

Show that f(0) and $f\left(\frac{\pi}{2}\right)$ differ in sign, and explain why f still does not vanish in $\left[0, \frac{\pi}{2}\right]$.

Sol. First of all we obtain expressions for f in $\left[0, \frac{\pi}{2}\right]$ in a form free from limits.

Since $\lim_{n \to \infty} x^{2n} = \begin{cases} 0 & \text{if} \quad 0 \le x < 1 \\ 1 & \text{if} \quad x = 1 \\ \infty & \text{if} \quad 1 < x \le \frac{\pi}{2} \end{cases}$

$$\therefore \quad f(x) = \lim_{x \to \infty} \frac{\log(2 + x) - x^{2n} \sin x}{1 + x^{2n}} = \begin{cases} \log(2 + x) \text{ if } 0 \le x < 1 \\ \frac{\log 3 - \sin 1}{2} \text{ if } x = 1 \\ \lim_{x \to \infty} \frac{x^{-2n} \log(2 + x) - \sin x}{x^{-2n} + 1} = -\sin x \text{ if } 1 < x \le \frac{\pi}{2} \end{cases}$$

$(\because \; x^{2n} \to \infty \Rightarrow x^{-2n} \to 0)$

Now $f(1) = \frac{1}{2}(\log 3 - \sin 1)$

$\lim_{x \to 1-}(x) = \lim_{x \to 1-} \log(2 + x) = \lim_{h \to 0+} \log(2 + 1 - h) = \log 3$

$\lim_{x \to 1+} f(x) = \lim_{x \to 1+}(-\sin x) = \lim_{h \to 0+} -\sin(1 + h) = -\sin 1$

Since $\lim_{x \to 1-} f(x)$ and $\lim_{x \to 1+} f(x)$ both exist, but are unequal, also neither of them is equal to f(1), therefore, f has a discontinuity of the first kind at x = 1 on both sides.

Now $f(0) = \log 2 > 0$ and $f\left(\frac{\pi}{2}\right) = \sin \frac{\pi}{2} = -1 < 0$

So that f(0) and $f\left(\frac{\pi}{2}\right)$ have opposite signs. Also, it is clear that f does not vanish anywhere in $\left[0, \frac{\pi}{2}\right]$.

The function f is not continuous on $\left[0, \frac{\pi}{2}\right]$, the point x = 1 being a point of discontinuity. This explains the reason why f does not vanish anywhere in $\left[0, \frac{\pi}{2}\right]$ even though f(0) and $f\left(\frac{\pi}{2}\right)$ are of opposite signs.

Remark. The above example shows that the hypothesis as well as the conclusion of the Intermediate value Theorem are not satisfied for the function f in $\left[0, \frac{1}{2}\pi\right]$.

Ex : Show that the function

$$\phi(x) = \lim_{x \to \infty} \frac{x^{2n+2} - \cos x}{x^{2n} + 1}$$

does not vanish anywhere in the interval [0, 2], though $\phi(0)$ and $\phi(2)$ differ in sign.

Hint. We have $f(x) = -\cos x$ for $0 \le x < 1$;

$$f(1) = \frac{1}{2}(1 - \cos x) \text{ and } \phi(x) = x^2 \text{ for } x > 1.$$

$$f(0) = -1 < 0, \ \phi(2) = 4 > 0.$$

Verify that ϕ is discontinuous at x = 1.

Ex : Given an example of a function which satisfies the conclusion but not the hypothesis of the Intermediate value theorem.

Sol. We defined a function f on [0, 1] as follows :

$$f(x) = \begin{cases} x, & \text{if } x \text{ is rational} \\ 1 - x, & \text{if } x \text{ is irrational} \end{cases}$$

Then f is not continuous in [0, 1] except at $x = \frac{1}{2}$

However, f takes every value between 0 and 1.

Uniform Continuity

Definition : A function f defined on an interval I is said to be **uniformly continuous in the interval** I, if for each $\varepsilon > 0$, there exists some $\delta > 0$ such that

$$| f(x_2) - f(x_1) | < \varepsilon, \text{ when } | x_2 - x_2 | < d \text{ and for all } x_1, x_2 \in I.$$

Remarks.

1. It may be noted that whereas continuity of a function is defined at a point, the uniform continuity of a function is defined in an interval. Even when we say that a function f is continuous in an interval it means that f is continuous at all points of the interval.
2. In case of continuity of a function at a point c, the choice of $\delta > 0$ depends upon $\varepsilon > 0$ and the point c. But in case of uniform continuity of a function in an interval, the choice of $\delta > 0$ depends only on $\varepsilon > 0$ and not on a pair of points of the given interval.
3. A function f is not **uniformly continuous** in an interval I, if there exists **some** $\varepsilon > 0$, such that **for any** $\delta > 0$, there exists a pair of points $x, y \in I$ for which $| f(x) - f(y) | \ge \varepsilon$, when $| x - y | < \delta$.

Ex : The function defined by $f(x) = x^2$ is uniformly continuous in (– 2, 2).

Sol. Let $\varepsilon > 0$ be any number and $x_1, x_2 \in (-2, 2)$. Then

$$| f(x_2) - f(x_1) | = | x_2^2 - x_1^2 | = | (x_2 - x_1)(x_2 + x_1) |$$
$$\le | x_2 - x_1 | (| x_2 | + | x_1 |)$$
$$< 4 | x_2 - x_1 | \qquad (\because \ x_1 < 2, x_2 < 2)$$

$\therefore$ $| f(x_2) - f(x_1) | < \varepsilon$, when $| x_2 - x_1 | < \varepsilon/4$

or $| f(x_2) - f(x_1) | < \varepsilon$, when $| x_2 - x_1 | < \delta \ (= \varepsilon/4) \ \forall \ x_1, x_2 \in (-2, 2)$.

Hence the function f is uniformly continuous in (– 2, 2).

Ex : Is the function $f(x) = \frac{x}{x+1}$ uniformly continuous for $x \in [0, 2]$? Justify your answer.

Sol. Let x, y be two arbitrary points $x = [0, 2]$. Then $x \geq 0, y \geq 0$

$\Rightarrow \quad x + 1 \geq 1$ and $y + 1 \geq 1 \Rightarrow (x + 1)(y + 1) \geq 1$ ...(1)

Now $|f(x) - f(y)| = \left|\frac{x}{x+1} - \frac{y}{y+1}\right| = \frac{|x-y|}{(x+1)(y+1)} \leq |x - y|$, by (1)

Let $\varepsilon > 0$ be given. Taking $\delta = \varepsilon$, we see that

$|f(x) - f(y)| < \varepsilon$, whenever $|x - y| < \delta$, $\forall$ $x, y \in [0, 2]$.

Hence f is uniformly continuous in [0, 2].

Ex : Show that the function f defined by $f(x) = x^3$ is uniformly continuous in the interval [0, 3].

Sol. Let $\varepsilon > 0$ and $x_1, x_2 \in [0, 3]$. $\therefore$ $x_1 \leq 3, x_2 \leq 3$.

$$\begin{aligned}\text{Now} \quad |f(x_2) - f(x_1)| &= | x_2^3 - x_1^3 | \\ &= | (x_2 - x_1)(x_2^2 + x_1^2 + x_1x_2) | \\ &\leq | x_2 - x_1 | [| x_1^2 | + | x_2^2 | + | x_1 x_2 |] \\ &= | x_2 - x_1 | [| x_1 |^2 + | x_2 |^2 + | x_1 | | x_2 |] \\ &\leq | x_2 - x_1 | [9 + 9 + 9]\end{aligned}$$

$\therefore \quad | f(x_2) - f(x_1) | \leq 27 | x_2 - x_1 |$

or $\quad | f(x_2) - f(x_1) | < \varepsilon$, whenever $| x_2 - x_1 | < \varepsilon/27$.

Thus for $\varepsilon > 0$, there exists $\delta = \varepsilon/27 > 0$ such that

$| f(x_2) - f(x_1) | < \varepsilon$, when $| x_2 - x_1 | < d$, $\forall$ $x_1, x_2 \in [0, 3]$.

Hence f is uniformly continuous in [0, 3].

Ex : Show that $f(x) = x^2$ is not uniformly continuous on $[0, \infty)$

Sol. Let $\varepsilon = 1/2$ and δ by any positive number. We can choose a positive integer n such that

$$n > \frac{1}{4\delta^2} \text{ or } \frac{1}{4n} < \delta^2. \quad ...(1)$$

Let $x_1 = \sqrt{n}$ and $x_2 = \sqrt{n+1} \in [0, \infty)$. Then

$| f(x_2) - f(x_1) | = | x_2^2 - x_1^2 | = | (n + 1) - n | = 1 > \varepsilon$,

and $| x_2 - x_1 | = |\sqrt{n+1} - \sqrt{n}| = \left|\frac{(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})}{\sqrt{n+1} + \sqrt{n}}\right|$

$$= \frac{n+1-n}{\sqrt{n+1}+\sqrt{n}} = \frac{1}{\sqrt{n+1}+\sqrt{n}} < \frac{1}{2\sqrt{n}} < \delta, \text{ by (1)}$$

$$(\because \sqrt{n+1} + \sqrt{n} > \sqrt{n} + \sqrt{n} = 2\sqrt{n})$$

The $| f(x_2) - f(x_1) | > \varepsilon$, when $| x_2 - x_1 | < \delta$.

Hence f is not uniformly continuous on $[0, \infty)$.

Ex : Let $f(x) = x^2$, $x \in \mathbb{R}$. Show that f is uniformly continuous on every closed and finite interval but is not uniformly continuous on $\mathbb{R}$.

Sol. Let [a, b] be any closed and finite interval. Let $x_1, x_2 \in [a, b]$. We have

$| f(x_2) - f(x_1) | = | x_2^2 - x_1^2 | = | x_2 - x_1 | | x_2 + x_1 | \leq | x_2 - x_1 | \{ | x_1 | + | x_2 | \}$

Let $k = \max \{ | x_1 |, | x_2 | \}$. Then $k > 0$.

$\therefore$ $|f(x_2) - f(x_1)| \le 2k\,|x_2 - x_1|$.

Let $\varepsilon > 0$ be given and let $\delta = \varepsilon/2k > 0$. Then

$|f(x_2) - f(x_1)| < \varepsilon$, whenever $|x_2 - x_1| < \delta$, $\forall\, x_1, x_2 \in [a, b]$.

Hence f is uniformly continuous on [a, b].

However, f is not uniformly continuous on $\mathbb{R}$.

Ex : Show that f(x) = 1/x is not uniformly continuous on (0, 1].

Sol. Let $\varepsilon = \frac{1}{2}$ and δ be any positive number. We can choose a positive integer n such that

$$n > \frac{1}{\delta}. \qquad \ldots(1)$$

Let $x_1 = \frac{1}{n}$ and $x_2 = \frac{1}{n+1} \in (0, 1]$. Then

$$|f(x_2) - f(x_1)| = \left|\frac{1}{x_2} - \frac{1}{x_1}\right| = |n + 1 - n| = 1 > \varepsilon,$$

and $|x_2 - x_1| = \left|\frac{1}{n+1} - \frac{1}{n}\right| = \frac{1}{n(n+1)} < \frac{1}{n} < \delta$ by (1)

$\therefore$ $|f(x_2) - f(x_1)| > \varepsilon$, when $|x_2 - x_1| < \delta$.

Hence 1/x is not uniformly continuous on (0, 1].

Ex : Show that f(x) = 1/x is not uniformly continuous on (0, 1), but it is uniformly continuous on [a, ∞), where a > 0.

Sol. By Example f(x) = 1/x is not uniformly continuous on (0, 1). Let x_1, x_2 Î [a, ∞) so that $x_1 \ge a > 0$. Now we show that f is not uniformly continuous in (0, ∞).

Let $\varepsilon = \frac{1}{2}$ and δ be any positive number. We can always choose a positive integer n such that

$$n > \frac{1}{2\delta} \quad \text{or} \quad \frac{1}{2n} < \delta. \qquad \ldots(1)$$

Let $x_1 = \frac{1}{\sqrt{n}}$ and $x_2 = \frac{1}{\sqrt{n+1}} \in (0, \infty)$. Then

$$|f(x_1) - f(x_2)| = \left|\frac{1}{x_1^2} - \frac{1}{x_2^2}\right| = |n - (n + 1)| = 1 > \varepsilon,$$

and $|x_1 - x_2| = \left|\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}\right| = \frac{|\sqrt{n+1} - \sqrt{n}|}{\sqrt{n}\sqrt{n+1}}$

$$= \frac{1}{\sqrt{n}\sqrt{n+1}(\sqrt{n+1} + \sqrt{n})}, \text{ on rationalising}$$

$$< \frac{1}{\sqrt{n}.2\sqrt{n}} \qquad [\because \sqrt{n}\sqrt{n+1} > \sqrt{n} \text{ and } \sqrt{n+1} + \sqrt{n} > 2\sqrt{n}]$$

$$= \frac{1}{2n} < \delta, \text{ by (1).}$$

Thus $|f(x_1) - f(x_2)| > \varepsilon$, when $|x_1 - x_2| < \delta$.

Hence f is not uniformly continuous on $(0, \infty)$.

Ex : Show that $f(x) = \sqrt{x}$ is uniformly continuous in $[0, 1]$.

Sol. Let $x, y \in [0, 1]$, where $x > y \geq 0$. Then

$$|f(x) - f(y)| = |\sqrt{x} - \sqrt{y}| = \left|\frac{(\sqrt{x} - \sqrt{y})(\sqrt{x} + \sqrt{y})}{\sqrt{x} + \sqrt{y}}\right|$$

$$= \frac{x - y}{\sqrt{x} + \sqrt{y}} \leq \frac{x - y}{\sqrt{x}} \leq \frac{x - y}{\sqrt{x - y}} = \sqrt{x - y}.$$

$\therefore \quad |f(x) - f(y)| \leq \sqrt{x - y}$.

Let $\varepsilon > 0$ be given. Then

$|f(x) - f(y)| < \varepsilon, \quad$ when $\sqrt{x - y} < \varepsilon$

or $\quad |f(x) - f(y)| < \varepsilon, \quad$ when $|x - y| < \delta = (= \varepsilon^2) \ \forall \ x, y \in [0, 1]$.

Hence f is uniformly continuous in $[0, 1]$.

Ex : Show that sin x is uniformly continuous on $[0, \infty)$.

Sol. Let $\varepsilon > 0$ be given and x, y be any two points in $[0, \infty)$. Let $f(x) = \sin x$. Then

$$|f(x) - f(y)| = |\sin x - \sin y|$$

$$= \left|2 \sin\frac{x - y}{2} \cos\frac{x + y}{2}\right|$$

$$= 2\left|\sin\frac{x - y}{2}\right|\left|\cos\frac{x + y}{2}\right|$$

$$\leq 2\left|\frac{x - y}{2}\right|.1 \quad [\text{ since } |\sin\theta| \leq |\theta|, |\cos\theta| \leq 1 \ \forall \ \theta]$$

$\therefore \quad |f(x) - f(y)| \leq |x - y|$

$\Rightarrow \quad |f(x) - f(y)| < \varepsilon$, when $|x - y| < \delta$, $(\delta = \varepsilon) \ \forall \ x, y \in [0, \infty)$

Hence $f(x) = \sin x$ is uniformly continuous on $[0, \infty)$.

Ex : Prove that $f(x) = \sin x^2$ is not uniformly continuous on $[0, \infty)$

Sol. Let $\varepsilon = 1/2$ and δ be any positive number. We can choose a positive integer n such that $n > \pi/\delta^2$. ...(1)

Let $\quad x_1 = \sqrt{\frac{n\pi}{2}}, x_2 = \sqrt{(n + 1)\frac{\pi}{2}} \in [0, \infty)$. Then

$$|f(x_2) - f(x_2)| = |\sin x_2^2 - \sin x_1^2|$$

$$= \left|\sin(n + 1)\frac{\pi}{2} - \sin\frac{n\pi}{2}\right|$$

$$= \begin{cases} |0 - (\pm 1)| = 1, \text{ if n is odd,} \\ |\pm 1 - 0| = 1, \text{ if n is even.} \end{cases}$$

$\therefore \quad | f(x_2) - f(x_1) | = 1 > \varepsilon,$

and $\quad |x_2 - x_1| = \left|\dfrac{x_2^2 - x_1^2}{x_2 + x_1}\right| = \dfrac{\pi/2}{\sqrt{(n+1)\frac{\pi}{2}} + \sqrt{\frac{n\pi}{2}}},$

$$< \frac{\pi}{2\left\{2\sqrt{\frac{n\pi}{2}}\right\}} < \frac{\pi}{\sqrt{n\pi}} = \sqrt{\frac{\pi}{n}} < \delta, \text{ by(1).}$$

$\therefore \quad | f(x_2) - f(x_1) | > \varepsilon$, when $| x_2 - x_1 | < \delta$.

Hence $f(x) = \sin x^2$ is not uniformly continuous on $[0, \infty)$.

Ex : Show that $f(x) = \sin\dfrac{1}{x}$ is not uniformly continuous on $(0, \infty)$.

Sol. Let $\varepsilon = \dfrac{1}{2}$ and δ be any positive number. We can choose a positive integer n such that $n\pi$ / $1/\delta$. ...(1)

Let $\quad x_1 = \dfrac{1}{n\pi}$ and $x_2 = \dfrac{2}{(2n+1)\pi} \in (0, \infty)$.

Then $\quad | f(x_1) - f(x_2) | = \left|\sin n\pi - \sin(2n+1)\dfrac{\pi}{2}\right| = 1 > \varepsilon,$

and $\quad | x_1 - x_2 | = \left|\dfrac{1}{n\pi} - \dfrac{2}{(2n+1)\pi}\right| = \dfrac{1}{n(2n+1)} < \dfrac{1}{n\pi} < \delta$, by (1).

$\therefore \quad | f(x_1) - f(x_2) | > \varepsilon$, when $| x_1 - x_2 | < \delta$.

Hence $f(x) = \sin\dfrac{1}{x}$ is not uniformly continuous on $(0, \infty)$.

Theorem : Every uniformly continuous function on an interval is continuous on that interval, but the converse is not true.

Proof. Let a function f be uniformly continuous on an interval I. Then for any $\varepsilon > 0$, there exists some $\delta > 0$ such that

$$| f(x) - f(y) | < \varepsilon, \quad \text{when} \quad | x - y | < \delta \quad \forall \; x, y \in I. \qquad \text{...(1)}$$

Let c be any point of I. Taking $y = c$ in (1), we obtain

$$| f(x) - f(c) | < \varepsilon, \quad \text{when} \quad | x - c | < \delta.$$

Thus f is continuous at the point c.

Since $c \in I$ is arbitrary, it follows that f is continuous at each point of I. Hence f is continuous on the interval I.

However, the converse of the theorem is not true.

A function which is continuous on an interval may not be uniformly continuous on that interval.

For example, the function $f(x) = 1/x \;\forall\; x \in (0, 1]$ is continuous on $(0, 1]$, but f is not uniformly continuous on $(0, 1]$.

However, if a function is continuous on a *closed interval*, then it is necessarily uniformly continuous on that closed interval as proved in the following :

Theorem : If a function f is continuous on a closed and bounded interval [a, b], then it is uniformly continuous on [a, b].

Proof. Let, if possible, f be not uniformly continuous on [a, b]. Then there exists some $\varepsilon > 0$ such that for any $\delta_n = 1/n$ ($n \in N$), there is a pair of elements $x_n, y_n \in [a, b]$ for which

$$|f(x_n) - f(y_n)| > \varepsilon, \quad \text{when} \quad |x_n - y_n| < \frac{1}{n}. \qquad ...(1)$$

Since $a \le x_n \le b$, $a \le y_n \le b$ for each n, the sequences $\langle x_n \rangle$ and $\langle y_n \rangle$ are bounded and so by Bolzano-Weierstrass theorem, they have limit points. Let α and β be limit points of $\langle x_n \rangle$, $\langle y_n \rangle$ respectively. It follows that α and β are limit points of [a, b]. Since a closed interval is a closed set and further a closed set contains all its limit points, so $\alpha, \beta \in [a, b]$.

Now a is a limit point of $\langle x_n \rangle$ implies that there exists a subsequence $\langle x_{n_k} \rangle$ of $\langle x_n \rangle$ such that

$$x_{n_k} \to \alpha. \qquad ...(2)$$

Similarly, $y_{n_k} \to \beta$, where $\langle y_{n_k} \rangle$ is a subsequence of $\langle y_n \rangle$. ...(3)

From (1), we conclude that

$$|f(x_{n_k}) - f(y_{n_k})| > \varepsilon, \quad \text{when} \quad |x_{n_k} - y_{n_k}| < \frac{1}{n_k}.$$

It is clear that $x_{n_k} \to y_{n_k} \to 0$ as $k \to \infty$

or $\qquad \lim x_{n_k} = \lim y_{n_k} \quad \Rightarrow \quad \alpha = \beta$, by (2) and (3).

But $\qquad |f(x_{n_k}) - f(y_{n_k})| > \varepsilon \quad \Rightarrow \quad \lim f(x_{n_k}) \ne \lim f(n_{n_k})$,

provided the two limits exist.

In particular, $\lim f(x_{n_k}) \ne f(\alpha)$ and $\lim (y_{n_k}) \ne f(\alpha)$.

Thus we have two sequences $\langle x_{n_k} \rangle, \langle y_{n_k} \rangle$ satisfying

$$x_{n_k} \to \alpha \text{ and } y_{n_k} \to \alpha,$$

but $\langle f(x_{n_k}) \rangle$ and $\langle f(y_{n_k}) \rangle$ do not converge to $f(\alpha)$, $\alpha \in [a, b]$.

It follows that f is not continuous at $\alpha \in [a, b]$, which is a contradiction to the given hypothesis. Hence f must be uniformly continuous in [a, b].

Ex : If f and g are uniformly continuous on an interval I, then prove that f + g is uniformly continuous on I.

Sol. Since f and g are uniformly continuous on I, for any $\varepsilon > 0$, $\exists$ some $\delta_1 > 0$ and $\delta_2 > 0$ such that $\forall$ x, y $\in$ I.

$$|f(x) - f(y)| < \varepsilon/2 \quad \text{when} \quad |x - y| < \delta_1, \qquad ...(1)$$

$$|g(x) - g(y)| < \varepsilon/2, \quad \text{when} \quad |x - y| < \delta_2. \qquad ...(2)$$

Let $\qquad \delta = \min(\delta_1, \delta_2)$. ...(3)

Now $\quad |(f + g)(x) - (f + g)(y)| = |\{f(x) - f(y)\} + \{g(x) - g(y)\}|$

$$\le |f(x) - f(y)| + |g(x) - g(y)|$$

$\therefore \quad |(f + g)(x) - (f + g)(y)| < \varepsilon/2 + \varepsilon/2 = \varepsilon$, when $|x - y| < \delta$

$\forall$ x, y $\in$ I ; using (1), (2) and (3).

Hence f + g is uniformly continuous in I.

Ex : If f is uniformly continuous in an interval I and $\langle x_n \rangle$ is a Cauchy sequence of elements in I, then $\langle f(x_n) \rangle$ is a Cauchy sequence.

Sol. Since f is uniformly continuous in I, for any $\varepsilon > 0$, there exists some $\delta > 0$ such that $\forall$ x, y $\in$ I,

$$| f(x) - f(y) | < \varepsilon, \text{ when } | x - y | < \delta. \qquad ...(1)$$

Since $\langle x_n \rangle$ is a Cauchy sequence in I, for $\varepsilon = \delta > 0$, $\exists$ a positive integer m such that

$$| x_n - x_m | < \delta \quad \forall \quad n \geq m. \qquad ...(2)$$

From (1) and (2), we obtain

$$| f(x_n) - f(x_m) | < \varepsilon \quad \forall \quad n \geq m. \qquad [\text{Take } x = x_n, y = x_m \text{ in } (1)]$$

Hence $\langle f(x_n) \rangle$ is a Cauchy sequence.

Ex : Prove that if f is uniformly continuous on a bounded interval I, then f is bounded on I.

Sol. Let, if possible, f be not bounded on I. Then for each n $\in$ N, there exists some $x_n \in$ I such that

$$| f(x_n) | > n \quad \forall \quad n \in N. \qquad ...(1)$$

Since I is a bounded interval and $x_n \in I \ \forall \ n \in N$, $\langle x_n \rangle$ is a bounded sequence in I. By Bolzano-Weierstrass Theorem, $\langle x_n \rangle$ of $\langle x_n \rangle$ has a limit point, say l. Then there exists a subsequence $\langle x_{n_k} \rangle$ of $\langle x_n \rangle$ such that $x_{n_k} \to l$ as $k \to \infty \Rightarrow \langle x_{n_k} \rangle$ is convergent and so $\langle x_{n_k} \rangle$ is a Cauchy sequence in I. Since f is uniformly continuous on I, $\langle f(x_{n_k}) \rangle$ is a Cauchy sequence and hence $\langle f(x_{n_k}) \rangle$ is bounded. From (1), we obtain

$$|f(x_{n_k})| > n_k \geq k \text{ for all positive integers k}$$

$\Rightarrow$ $\langle f(x_{n_k}) \rangle$ is not bounded, which is a contradiction.

Hence f is bounded on I.

Ex : Justify with an example that the product of two uniformly continuous functions may not be uniformly continuous.

Sol. We shall prove that f(x) = x + 1 is uniformly continuous on $\mathbb{R}$. Let $\varepsilon > 0$ be given. Then for any x, y $\in \mathbb{R}$; we have

$$| f(x) - f(y) | = | (x + 1) - (y + 1) | = | x - y |.$$

$$\therefore \quad | f(x) - f(y) | < \varepsilon, \text{ when } | x - y | < \delta = \varepsilon, \forall\ x, y \in \mathbb{R}.$$

Thus f(x) = x + 1 is uniformly continuous on $\mathbb{R}$.

Similarly, g(x) = x − 1 is uniformly continuous on $\mathbb{R}$.

We shall now prove that f(x) g(x) = $x^2 - 1$ is not uniformly continuous on $\mathbb{R}$. For and $\delta > 0$, let

$$x_1 = \frac{1}{\delta} \quad \text{and} \quad x_2 = \frac{1}{\delta} + \frac{\delta}{2}. \quad \text{Then} \quad | x_1 - x_2 | = \frac{\delta}{2} < \delta \quad \text{and}$$

$$| f(x_1) - f(x_2) | = |x_1 - x_2^2 | = \left| \frac{1}{\delta^2} - \left(\frac{1}{\delta} + \frac{\delta}{2} \right)^2 \right| = 1 + \frac{\delta^2}{4} > 1.$$

If we choose $\varepsilon = 1$, then $| f(x_1) - f(x_2) | \leq \varepsilon$ for $| x_1 - x_2 | < \delta$. Hence the product $x^2 - 1$ of two uniformly continuous functions x + 1 and x − 1 on $\mathbb{R}$ is not uniformly continuous on $\mathbb{R}$.

6. DIFFERENTIABILITY OF A FUNCTION OF ONE VARIABLE

Definition : A function f is differentiable at an interior point x_0 of its domain if the difference quotient

$$\frac{f(x)-f(x_0)}{x-x_0}, \quad x \neq x_0,$$

approaches a finite limit as x approaches x_0, in which case the limit is called the *derivative* of f at x_0, and is denoted by $f'(x_0)$; thus

$$f'(x_0)=\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}. \qquad ...(1)$$

It is sometimes convenient to let $x = x_0 + h$ and write (1) as

$$f'(x_0)=\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}.$$

If f is defined on an open set S, we say that f is *differentiable* on S if f is differentiable at every point of S. If f is differentiable on S, then f' is a function on S. We say that f is *continuously differentiable* on S if f' is continuous on S. If f is differentiable on a neighborhood of x_0, it is reasonable to ask if f' is differentiable at x_0. If so, we denote the derivative of f' at x_0 be $f''(x_0)$. This is the *second derivative* of f at x_0, and it is also denoted by $f^{(2)}(x_0)$. Continuing inductively, if $f^{(n-1)}$ is defined on a neighborhood of x_0, then the nth *derivative* of f at x_0, denoted by $f^{(n)}(x_0)$, is the derivative of $f^{(n-1)}$ at x_0. For convenience we defined the *zeroth derivative* of f to be f itself; thus

$$f^{(0)} = f.$$

We assume that you are familiar with the other standard notations for derivatives; for example,

$$f^{(2)} = f'', \quad f^{(3)} = f''',$$

and so on, and

$$\frac{d^n f}{dx^n} = f^{(n)}.$$

Let f be a real valued function defined on an interval [a, b] i.e.,

$f : [a, b] \to \mathbb{R}$

Let $a < c < b$.

Definition : The **left hand derivative** of a function f at x = c, denoted by L f'(c) or f' (c – 0) is defined as

$$L\, f'(c) = \lim_{x\to c-}\frac{f(x)-f(c)}{x-c}, \text{ provided the limit exists.}$$

Definition : The **right hand derivative** if a function f at x = c, denoted by R f' (c) or f' (c + 0), is defined as

$$R\, f'(c) = \lim_{x\to c+}\frac{f(x)-f(c)}{x-c}, \text{ provided the limit exists.}$$

Definition : If L f' (c) = R f' (c) then f is differentiable at x = c.

$$f'(c)=\lim_{x\to c}\frac{f(x)-f(c)}{x-c}$$

Definition : A function f defined on an interval [a, b] is said to be **differentiable in the interval** [a,b], if f is differentiable at all points of the interval [a, b].

Ex : If n is a positive integer and

$$f(x) = x^n,$$

then

$$\frac{f(x)-f(x_0)}{x-x_0} = \frac{x^n - x_0^n}{x-x_0} = \frac{x-x_0}{x-x_0}\sum_{k=0}^{n-1} x^{n-k-1}x_0^k,$$

so

$$f'(x_0) = \lim_{x\to x_0}\sum_{k=0}^{n-1} x^{n-k-1}x_0^k = nx_0^{n-1}.$$

Since this holds for every x_0, we drop the subscript and write

$$f'(x) = nx^{n-1} \quad \text{or} \quad \frac{d}{dx}(x^n) = nx^{n-1}.$$

Interpretations of the Derivative

If f(x) is the position of a particle at time $x \neq x_0$, the difference quotient

$$\frac{f(x)-f(x_0)}{x-x_0} \qquad \text{...(1)}$$

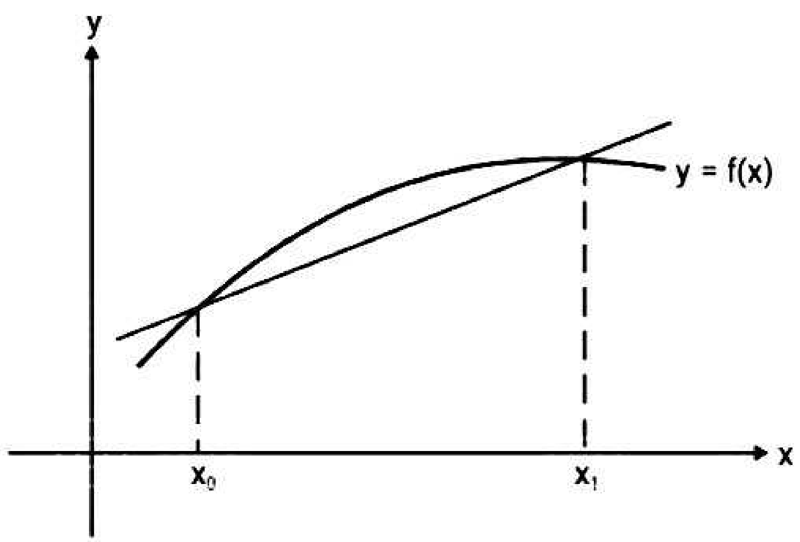
is the average velocity of the particle between times x_0 and x. As x approaches x_0, the average applies to shorter and shorter intervals. Therefore, it makes sense to regard the limit (1), if it exists, as the particle's *instantaneous velocity at time* x_0. This interpretation may be useful even if x is not time, so we often regard $f'(x_0)$ as the *instantaneous rate of change of f(x) at* x_0, regardless of the specific nature of the variable x. The derivative also has a geometric interpretation. The equation of the line through two points $(x_0, f(x_0))$ and $(x_1, f(x_1))$ on the curve y = f(x) (figure) is

$$y = f(x_0) + \frac{f(x_1)-f(x_0)}{x_1 - x_0}(x - x_0).$$

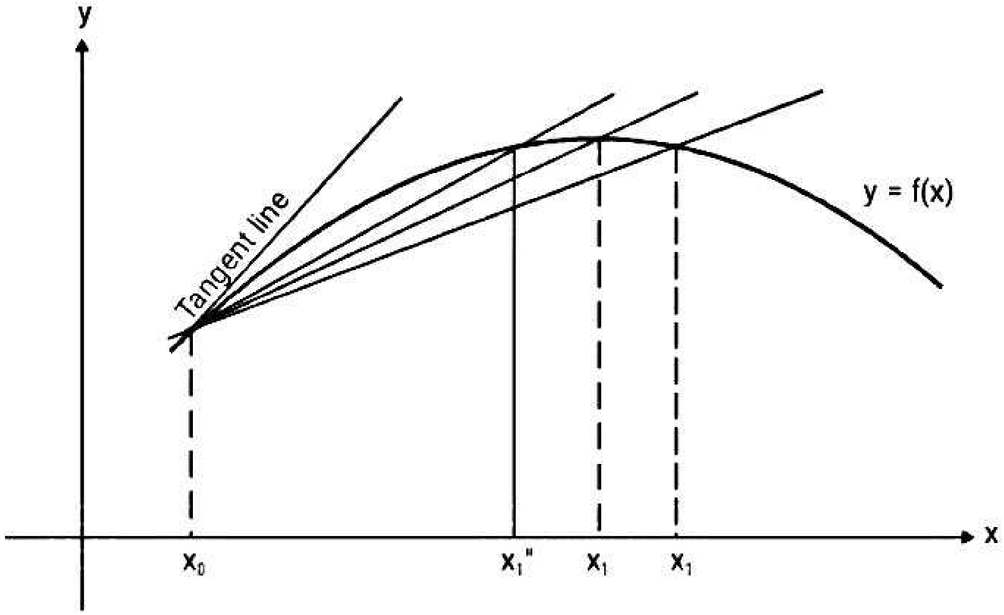
Varying x_1 generates lines through $(x_0, f(x_0))$ that rotate into the line

$$y = f(x_0) + f'(x_0)(x - x_0)$$

as x_1 approaches x_0. This the *tangent* to the curve y = f(x) at the point $(x_0, f(x_0))$. Figure depicts the situation for various values of x_1.



Figure



Figure

Here is a less intuitive definition of the tangent line: If the function

$$T(x) = f(x_0) + m(x - x_0) \quad ...(2)$$

approximates f so well near x_0 such that

$$\lim_{x \to x_0} \frac{f(x) - T(x)}{x - x_0} = 0,$$

we say that the line y = T(x) is *tangent to the curve* y = f(x) at $(x_0, f(x_0))$.

This tangent line exists if and only if $f'(x_0)$ exists, in which case m is uniquely determined by $m = f'(x_0)$. Thus, (2) is the equation of the tangent line.

We will use the following lemma to study differentiable functions.

Lemma : If f is differentiable at x_0, then

$$f(x) = f(x_0) + [f'(x_0) + E(x)](x - x_0),$$

where E is defined on a neighborhood of x_0 and

$$\lim_{x \to x_0} E(x) = E(x_0) = 0.$$

Theorem : Differentiability implies Continuity.

Proof. Let f be differentiable at x = c, so that

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - x}.$$

Now $\lim_{x \to c} [f(x) - f(c)] = \lim_{x \to c} \left[\frac{f(c) - f(c)}{x - c} \times (x - c)\right]$

$$= \lim_{x \to c} \frac{f(c) - f(c)}{x - c} \cdot \lim_{x \to c} (x - c)$$

$$= f'(c) \times 0 = 0.$$

$\therefore \lim_{x \to c} f(x) = f(c)$. Hence f is continuous at x = c.

The converse may not be true.

The function f(x) = | x | is continuous at x = 0 but not differentiable at x = 0.

We know $|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$

Now $\lim\limits_{x\to 0-} f(x) = \lim\limits_{x\to 0} (-x) = 0,$

$\lim\limits_{x\to 0+} f(x) = \lim\limits_{x\to 0} (-x) = 0.$ Also f(0) = 0.

$\therefore \quad \lim\limits_{x\to 0-} f(x) = \lim\limits_{x\to 0-} f(x) = f(0).$

Hence f is continuous at x = 0.

Now $\dfrac{f(x)-f(0)}{x-0} = \dfrac{|x|-0}{x} = \dfrac{|x|}{x}.$

$\therefore \quad L\,f'(0) = \lim\limits_{x\to 0-} \dfrac{f(x)-f(0)}{x-0} = \lim\limits_{x\to 0} \dfrac{(-x)}{x} = -1.$

$R\,f'(0) = \lim\limits_{x\to 0+} \dfrac{f(x)-f(0)}{x-0} = \dfrac{x}{x} = 1.$

Thus L f' (0) ≠ R f' (0) implies f is not differentiable at x = 0.

Ex : The function

$$f(x) = \begin{cases} x\sin\dfrac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

is continuous at x = 0 but not differentiable at x = 0.

Sol. We have $\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0}\left(x\sin\dfrac{1}{x}\right) = 0.$

Thus $\lim\limits_{x\to 0} f(x) = f(0)$ and so f is continuous at x = 0.

Now $\lim\limits_{x\to 0} \dfrac{f(x)-f(0)}{x-0} = \lim\limits_{x\to 0} \sin\dfrac{1}{x}$, which does not exist.

Hence f is not differentiable at x = 0.

Ex : Show that the function f(x) = x | x | is differentiable at the origin.

Sol. We have

$$L\,f'(0) = \lim_{x\to 0-} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} \frac{x(-x)}{x} = 0,$$

$$R\,f'(0) = \lim_{x\to 0+} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} \frac{x(x)}{x} = 0.$$

$\therefore \quad$ L f' (0) = R f' (0) = 0.

Hence f' (0) = 0.

Ex : Let f be defined by setting

$$f(x) = \begin{cases} x, & \text{if } x \leq 0 \\ 0, & \text{if } x > 0. \end{cases}$$

Show that f is differentiable at all points except x = 0.

Sol. We have

$$Lf'(0) = \lim_{x \to 0-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{x - 0}{x} = 1,$$

$$Rf'(0) = \lim_{x \to 0+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{0 - 0}{x} = 0.$$

Since $L\,f'(0) \neq R\,f'(0)$, f is not differentiable at x = 0.

Obviously $f'(x) = 1$, if $x < 0$ and $f'(x) = 0$, if $x > 0$.

Hence f is differentiable at all points except x = 0.

Ex : Discuss the differentiability of the function

$$f(x) = \begin{cases} 2x - 3, & 0 \le x \le 2 \\ x^2 - 3, & 2 < x \le 4 \end{cases}$$

at x = 2, 4.

Sol. We have

$$L\,f'(2) = \lim_{x \to 2-} \frac{f(x) - f(2)}{x - 2} = \lim_{x \to 2} \frac{(2x - 3) - 1}{x - 2} = 2,$$

$$R\,f'(2) = \lim_{x \to 2+} \frac{f(x) - f(2)}{x - 2} = \lim_{x \to 2} \frac{(x^2 - 3) - 1}{x - 2} = \lim_{x \to 2} (x + 2) = 4.$$

Since $L\,f'(2) \neq R\,f'(2)$, f is not differentiable at x = 2.

$$L\,f'(4) = \lim_{x \to 4-} \frac{f(x) - f(4)}{x - 4} = \lim_{x \to 4} \frac{(x^2 - 3) - 13}{x - 4} = \lim_{x \to 4} (x + 4) = 8.$$

Since the given function is defined on [0, 4], therefore $f'(4) = L\,f'(4) = 8$. Hence f is differentiable at x = 4.

Ex : Discuss the differentiability of the function

$$f(x) = \begin{cases} x & \text{if } x < 1 \\ 2 - x & \text{if } 1 \le x \le 2 \\ -2 + 3x - x^2 & x > 2 \end{cases}$$

at x = 1, 2.

Sol. $L\,f'(1) = \lim_{x \to 1^-} \frac{f(x) - f(2)}{x - 2} = \lim_{x \to 1^-} \frac{(2 - x) - 0}{x - 2} = -1,$

$$R\,f'(1) = \lim_{x \to 1^+} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1^+} \frac{(2 - x) - 1}{x - 1} = -1,$$

$\therefore$ $\quad L\,f'(1) \neq R\,f'(1).$

Hence f is not differentiable at x = 1.

Now $\quad L\,f'(2) = \lim_{x \to 2^-} \frac{f(x) - f(2)}{x - 2} = \lim_{x \to 2^-} \frac{(2 - x) - 0}{x - 2} = -1,$

$$R\,f'(2) = \lim_{x \to 2^-} \frac{f(x) - f(2)}{x - 2} = \lim_{x \to 2^-} \frac{(-2 + 3x - x^2) - 0}{x - 2}$$

$$= \lim_{x \to 2^-} \frac{-(x - 1)(x - 2)}{(x - 2)} = \lim_{x \to 2^-} \{-(x - 1)\} = -1.$$

$\therefore \quad Lf'(2) = Rf'(2) = -1.$

Hence f is differentiable at x = 2 and f' (2) = –1.

Ex : Prove that the function defined as

$$f(x) = x \cdot \frac{e^{1/x} - 1}{e^{1/x} + 1}, \text{ if } x \neq 0$$

$$= 0, \text{ if } x = 0$$

is not differentiable at x = 0. Further show that f is continuous at x = 0.

Sol. We have

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{f(x)}{x} \qquad (\because f(0) = 0)$$

$$= \lim_{x \to 0} \frac{e^{1/x} - 1}{e^{1/x} + 1}, \text{ which does not exist.}$$

Thus f is not differentiable at x = 0.

Now we show that f is continuous as x = 0.

$$\lim_{x \to 0-} f(x) = \lim_{x \to 0-} \frac{x(e^{1/x} - 1)}{(e^{1/x} + 1)} = \frac{0.(0-1)}{0+1} = 0,$$

$$\left(\because \lim_{x \to 0-} e^{1/x} = 0\right)$$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{x(1 - e^{-1/x})}{(1 + e^{-1/x})} = \frac{0.(1-0)}{1+0} = 0$$

$$\left(\because \lim_{x \to 0+} e^{-1/x} = 0\right)$$

Also f(0) = 0. Hence f is continuous at x = 0.

Ex : Examine the function f where

$$f(x) = x\frac{e^{1/x} - e^{-1/x}}{e^{1/x} + e^{-1/x}}, x \neq 0, f(0) = 0$$

as regards continuity and differentiability at the origin.

Sol. We have

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{f(x)}{x} = \lim_{x \to 0} \frac{e^{1/x} - e^{-1/x}}{e^{1/x} + e^{-1/x}},$$

which does not exist.

Thus f is not differentiable at x = 0.

Now $$\lim_{x \to 0-} f(x) = \lim_{x \to 0-} \frac{x(e^{1/x} - e^{-1/x})}{(e^{1/x} + e^{-1/x})} = \lim_{x \to 0-} \frac{x(e^{1/x}.e^{1/x} - 1)}{(e^{1/x}.e^{1/x} + 1)}$$

$$= \frac{0.(0-1)}{0+1} = 0,$$

$$\lim_{x \to 0+} f(x) = \lim_{x \to 0+} \frac{x(1 - e^{-1/x}.e^{-1/x})}{(1 + e^{-1/x}.e^{-1/x})} = \frac{0.(1-0)}{1+0} = 0.$$

Also f(0) = 0. Hence f is continuous at x = 0.

Ex : If $f(x) = x^2 \sin(1/x)$ when $x \neq 0$ and $f(0) = 0$, show that f is differentiable for every value of x but the derivative is not continuous for x = 0.

Sol. We have

$$\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0}\frac{f(x)}{x}=\lim_{x\to 0} x\sin\frac{1}{x}=0 \qquad ...(1)$$

So $f'(0) = 0$, which means f is differentiable at x = 0.

For $x \neq 0$, $\quad f'(x) = 2x \sin(1/x) + x^2 \cos(1/x)(-1/x^2)$

$$f'(x) = 2x \sin(1/x) - \cos(1/x). \qquad ...(2)$$

Hence f is differentiable for every value of x.

Now we discuss the continuity of f' at x = 0. We have

$$\lim_{x\to 0} f'(x)=2\lim_{x\to 0} x\sin\frac{1}{x}-\lim_{x\to 0}\cos\frac{1}{x}=0-\lim_{x\to 0}\cos\frac{1}{x}.$$

Since $\lim\limits_{x\to 0}\cos\frac{1}{x}$ does not exist, it follows that f' is not continuous at x = 0.

Ex : If $f(x) = \begin{cases} x^3\sin\frac{1}{x}, & x\neq 0 \\ 0, & x=0, \end{cases}$

prove that f(x) has a derivative at x = 0 and that f(x) and f'(x) are continuous at x = 0.

Sol. We know $\lim\limits_{x\to 0} x\sin\frac{1}{x} = 0$. Also $f(0) = 0$.

$$f(0) = \lim_{x\to 0}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0}\frac{x^3\sin(1/x)}{x}$$

$$=\lim_{x\to 0} x^2\sin(1/x)=\lim_{x\to 0}\{x(x\sin(1/x))\}$$

$$=\lim_{x\to 0} x\times\lim_{x\to 0} x\sin(1/x)=0\times 0=0.$$

$\therefore \quad f'(0) = 0$ and so f(x) is differentiable at x = 0. $\qquad$ ...(1)

Now $\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0} x^3 \sin(1/x) = \lim\limits_{x\to 0} x^2\{x\sin(1/x)\}$

$$=\lim_{x\to 0} x^2\times\lim_{x\to 0} x\sin(1/x)=0\times 0=0=f(0).$$

So $\lim\limits_{x\to 0} f(x) = f(0)$ implies that f(x) is continuous at x = 0.

Now $\quad f'(x) = 3x^2\sin\frac{1}{x} + x^3\cos\left(\frac{1}{x}\right)\left(-\frac{1}{x^2}\right)$, if $x \neq 0$

$$=3x\left(x\sin\frac{1}{x}\right)-x\cos\frac{1}{x}.$$

$$\therefore \quad \lim_{x\to 0} f'(x)=\lim_{x\to 0}(3x)\lim_{x\to 0}\left(x\sin\frac{1}{x}\right)-\lim_{x\to 0} x\cos\frac{1}{x}$$

$$= 0\times 0 - 0 = 0 = f'(0), \text{ by (1)}$$

So $\lim\limits_{x\to 0} f'(x) = f'(0)$ implies that f'(x) is continuous at x = 0.

Ex : Let f be defined on $\mathbb{R}$ by setting

$f(x) = x^4 \sin(1/x)$, if $x \neq 0$ and $f(0) = 0$.

Show that $f''(0)$ exists but f'' is not continuous at $x = 0$.

Sol. We have

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{x^4 \sin(1/x)}{x} = \lim_{x \to 0} x^3 \sin(1/x)$$

$$= \lim_{x \to 0} [x^2 . x \sin(1/x)] = 0 \times 0 = 0 \qquad [\because \lim_{x \to 0} x \sin(1/x) = 0]$$

Thus $f'(0) = 0$. Now

$f'(x) = 4x^3 \sin(1/x) - x^2 \cos(1/x), x \neq 0$.

$\therefore$ $f''(x) = [12 x^2 \sin(1/x) - 4x \cos(1/x)] - [2x \cos(1/x) + \sin(1/x)]$

or $f''(x) = 12 x^2 \sin(1/x) - 6x \cos(1/x) - \sin(1/x)$.

Now $\lim_{x \to 0} f''(x) = 12 \lim_{x \to 0} x. \lim_{x \to 0} \{\sin(1/x)\} - 6 \lim_{x \to 0} \{x \cos(1/x)\} - \lim_{x \to 0} \sin(1/x)$

$$= - \lim_{x \to 0} \sin(1/x), \text{ which does not exist}$$

Thus f'' is not continuous at $x = 0$. Now

$$f''(0) = \lim_{x \to 0} \frac{f'(x) - f'(0)}{x - 4}$$

$$= \lim_{x \to 0} \frac{[4x^3 \sin(1/x) - x^2 \cos(1/x)] - 0}{x}$$

$$= \lim_{x \to 0} [4x \{(x \sin(1/x)\} - \{x \cos(1/x)\}] = 0.$$

Hence $f''(0)$ exists but f'' is not continuous at $x = 0$.

Ex : Show that the function f defined by

$$f(x) = \begin{cases} x\left[1 + \frac{1}{3}\sin(\log x^2)\right], & x \neq 0 \\ 0, & x = 0 \end{cases}$$

is continuous at $x = 0$ but not differentiable at $x = 0$.

Sol. We have

$$| f(x) - f(0) | = | f(x) | = | x | \left|1 + \frac{1}{3}\sin(\log x^2)\right|$$

$$\leq |x| . \left[1 + \frac{1}{3} |\sin(\log x^2)|\right]$$

$$\leq |x| . \left(1 + \frac{1}{3}\right) = \frac{4}{3} |x - 0|, \text{ since } |\sin\theta| \leq 1 \ \forall \ \theta.$$

Let $\varepsilon > 0$ be given and $\delta = 3\varepsilon/4$. Then

$| f(x) - f(0) | < \varepsilon$, if $| x - 0 | < \delta$. Hence f is continuous at $x = 0$.

Now $\quad R\ f'(0) = \lim_{x \to 0+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0+} \frac{f(x)}{x}$

$$= \lim_{x \to 0+} \left[1 + \frac{1}{3}\sin(\log x^2)\right]$$

$$= 1 + \frac{1}{3}\lim_{x \to 0+} \sin(\log x^2), \text{ which does not exist.}$$

Hence f is not differentiable at x = 0.

Ex : Show that the function f defined as

$$f(x) = |x - 1| + |x + 1| \quad \forall \quad x \in \mathbb{R}$$

is not differentiable at the points x = –1 and x = 1, and is differentiable at every other point.

Sol. We have $|x - 1| = x - 1$, if $x \geq 1$

and $\quad |x - 1| = -(x - 1)$, if $x < 1$.

Also $\quad |x + 1| = x + 1$, if $x \geq -1$ and $|x + 1| = -(x + 1)$, if $x < 1$.

$$\therefore \quad f(x) = \begin{cases} -(x-1)-(x+1) = -2x, & \text{if} \quad x < -1 \quad \ldots(1) \\ (x+1)-(x-1) = 2, & \text{if} \quad -1 \leq x < 1, \quad \ldots(2) \\ (x-1)+x+1 = 2x, & \text{if} \quad x \geq 1. \quad \ldots(3) \end{cases}$$

Let $\quad L\ f'(-1) = \lim_{\substack{x \to -1 \\ (x < -1)}} \frac{f(x) - f(-1)}{x - (-1)}$

$$= \lim_{x \to -1} \frac{-2x - 2}{x + 1} = \lim_{x \to -1} \frac{-2(x+1)}{x+1} = -2, \text{ using (1)}$$

$$R\ f'(-1) = \lim_{\substack{x \to -1 \\ (x > -1)}} \frac{f(x) - f(-1)}{x - (-1)} = \lim_{x \to -1} \frac{2 - 2}{x + 1} = 0, \text{ using (2).}$$

Since $L\ f'(-1) \neq R\ f'(-1)$, f is not differentiable at x = –1.

Now $\quad L\ f'(1) = \lim_{\substack{x \to 1 \\ (x < 1)}} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1} \frac{2 - 2}{x - 1} = 0, \text{ using (2)}$

$$R\ f'(1) = \lim_{\substack{x \to 1 \\ (x > 1)}} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1} \frac{2x - 2}{x - 1} = 2, \text{ using (3)}$$

Since $L\ f'(1) \neq R\ f'(1)$, f is not differentiable at x = 1.

From (1) $\quad f'(x) = -2 \ \forall \ x < -1$.

From (2), $\quad f'(x) = 0$, if $-1 < x < 1$.

From (3), $\quad f'(x) = 2 \ \forall \ x > 1$.

Algebra of Derivatives

Theorem: If f and g be two functions which are defined on [a, b] and differentiable at any point $c \in [a, b]$, then

$$(f + g)'(c) = f'(c) + g'(c).$$

Further for each real number k, the function kf is also differentiable at c and

$$(kf)'(c) = kf'(c).$$

Proof. (i) Since f and g are differentiable at c, therefore

$$\lim_{x \to c} \frac{f(x)-f(c)}{x-c} = f'(c) \text{ and } \lim_{x \to c} \frac{g(x)-g(c)}{x-c} = g'(c) \qquad ...(1)$$

Consider $\lim_{x \to c} \frac{(f+g)(x)-(f+g)(c)}{x-c}$

$$= \lim_{x \to c} \frac{[f(x)+g(x)]-[f(c)+g(c)]}{x-c}$$

$$= \lim_{x \to c} \frac{f(x)-f(c)}{x-c} + \lim_{x \to c} \frac{g(x)-g(c)}{x-c}$$

= f' (c) + g' (c), using (1).

Hence f + g is differentiable at c and (f + g)' (c) = f' (c) + g' (c).

(ii) Now $\lim_{x \to c} \frac{(kf)(x)-(kf)(c)}{x-c} = \lim_{x \to c} \left[k\left(\frac{f(x)-f(c)}{x-c}\right)\right]$

$$= k \lim_{x \to c} \frac{f(x)-f(c)}{x-c} = kf'(c).$$

Hence kf is differentiable at c and (kf)' (c) = kf' (c).

Theorem : If f and g are two differentiable functions at x = c, then fg also differentiable at x = c and

$$(fg)'(c) = f(c)\, g'(c) + g(c)\, f'(c).$$

Proof. Since f and g are differentiable at x = c, therefore

$$f'(c) = \lim_{x \to c} \frac{f(x)-f(c)}{x-c} \text{ and } g'(c) = \lim_{x \to c} \frac{g(x)-g(c)}{x-c} \qquad ...(1)$$

Since f and g are differentiable at x = c, so f and g are continuous and x = c. Thus

$$\lim_{x \to c} f(x) = f(c), \ \lim_{x \to c} g(x) = g(c). \qquad ...(2)$$

Now $\lim_{x \to c} \frac{(fg)(x)-(fg)(c)}{x-c} = \lim_{x \to c} \frac{f(x)g(x)-f(c)g(c)}{x-c}$

$$= \lim_{x \to c} \frac{f(x)[g(x)-g(c)]+g(c)[f(x)-f(c)]}{x-c}$$

$$= \lim_{x \to c} f(x) . \lim_{x \to c} \frac{g(x)-g(c)}{x-c} + g(c) \ . \ \lim_{x \to c} \frac{f(x)-f(c)}{x-c}$$

= f(c) g'(c) + g(c) f'(c), using (1) and (2).

Hence fg is differentiable at c and

$$(fg)'(c) = f(c)\, g'(c) + g(c)\, f'(c).$$

Theorem : (i) If a function f is differentiable at c and if f(c) ≠ 0, then 1/f is also differentiable at c and

$$\left(\frac{1}{f}\right)'(c) = \frac{-f'(c)}{[f(c)]^2}.$$

(ii) If f and g be two functions on [a, b] and differentiable at any point $c \in [a, b]$ and if $g(c) \neq 0$, then f/g is also differentiable at c and

$$\left(\frac{f}{g}\right)'(c) = \frac{g(c)\,f'(c) - f(c)\,g'(c)}{[g(c)]^2}.$$

Proof. (i) Since f is differentiable at c, so f is continuous at c.

Thus $\lim_{x \to c} f(x) = f(c)$. Also $f'(c) = \lim_{x \to c} \frac{f(c) - f(c)}{x - c}$

Now $$\frac{\frac{1}{f(x)} - \frac{1}{f(c)}}{x - c} = -\frac{f(x) - f(c)}{x - c} \cdot \frac{1}{f(x)} \cdot \frac{1}{f(c)}$$

$$\therefore \quad \lim_{x \to c} \frac{\frac{1}{f(x)} - \frac{1}{f(c)}}{x - c} = -\lim_{x \to c} \frac{f(x) - f(c)}{x - c} \cdot \lim_{x \to c} \frac{1}{f(x)} \cdot \frac{1}{f(c)}$$

$$= -f'(c) \cdot \frac{1}{f(c)} \cdot \frac{1}{f(c)} = -\frac{f'(c)}{\{f(c)\}^2}.$$

Hence 1/f is differentiable at c and

$$\left(\frac{1}{f}\right)'(c) = -\frac{f'(c)}{[f(c)]^2}. \qquad \text{...(2)}$$

(ii) Since g is differentiable at c and $g(c) \neq 0$. So 1/g is differentiable at c. By Theorem , we see that

f and $\frac{1}{g}$ and differentiable at c $\Rightarrow$ f. $\frac{1}{g}$ is differentiable at c.

Hence f/g is differentiable at c. Again by Theorem, we have

$$\left(\frac{f}{g}\right)'(c) = f'(c) \cdot \frac{1}{g(c)} + f(c) \cdot \left(\frac{1}{g}\right)'(c)$$

$$= f'(c) \cdot \frac{1}{g(c)} - f(c) \frac{g'(c)}{\{g(c)\}^2}, \text{ using (1)}$$

Hence $\left(\frac{f}{g}\right)'(c) = \frac{g(c) f'(c) - f(c)\, g'(c)}{\{g(c)\}^2}$

Theorem : (Chain Rule)

Let $f : I \to \mathbb{R}$ and $g : J \to \mathbb{R}$ be real-valued functions defined on the intervals I and J, respectively such that $f(I) \subset J$. Let f be differentiable at $c \in I$ and g be differentiable at f(c). Then gof is differentiable at c and

$$(gof)'(c) = g'(f(c)).f'(c).$$

Theorem (Inverse Function Theorem for Derivatives)

Let f be a continuous one-to-one function defined on an interval and let f be differentiable at x_0 with $f'(x_0) \neq 0$. Then the inverse of the function f is differentiable at $f(x_0)$ and its derivative at $f(x_0)$ is $1/f'(x_0)$.

The proof is omitted.

Ex : Let f and g be two functions having the same domain D. If fg is differentiable at $x_0 \in D$, is it necessary that f and g are both differentiable at x_0? Give argument in support of your answer.

Sol. The function f and g may not be both differentiable at x_0.

Let $f(x) = |x|$ and $g(x) = -|x| \ \forall \ x \in \mathbb{R}$.

Then $(fg)(x) = f(x)\,g(x) = -|x|^2 = -x^2$. $\quad (\because \ |x|^2 = x^2)$

Obviously, fg is differentiable at x = 0, since

$(fg)'(x) = -2x$ and $(fg)'(0) = 0$.

But f and g are both not differentiable at 0.

Ex : Let f and g be two functions having the same domain D. If f ± g and f/g are differentiable at $x_0 \in D$, is it necessary that f and g are both differentiable at x_0?

Sol. (i) The functions f and g may not be both differentiable at x_0.

Let $f(x) = x \sin\frac{1}{x}$, $x \neq 0$ and $f(0) = 0$.

Let $g(x) = -f(x) \ \forall \ x \in \mathbb{R}$.

Then $(f + g)(x) = f(x) + g(x) = 0 \ \forall \ x \in \mathbb{R}$.

$\therefore$ f + g is differentiable at 0 and $(f + g)'(0) = 0$.

But f and g are both not differentiable at 0.

(ii) Let $f(x) = g(x) = |x| \ \forall \ x \in \mathbb{R}$.

Then $(f - g)(x) = f(x) - g(x) = 0 \ \forall \ x \in \mathbb{R}$.

Obviously, f − g is differentiable at x = 0 and $(f - g)'(0) = 0$.

But f and g are both not differentiable at x = 0.

(iii) Let $f(x) = x \sin\frac{1}{x}$, $x \neq 0$; $f(0) = 0$. Let $g(x) = \frac{1}{x}$, $x \neq 0$; $g(0) = 1$.

Then f and g are both not differentiable at 0. But

$$\left(\frac{f}{g}\right)(x) = \begin{cases} x^2 \sin\frac{1}{x}, & x \neq 0 \\ 0 \quad , & x = 0 \end{cases}$$

is differentiable at x = 0.

Ex : Let $f(x) = |x|$ and $g(x) = 3|x|$. Show that f and g are not differentiable at the origin. But $\lim_{x \to 0} \frac{f(x)}{g(x)}$ exists and is equal to $\lim_{x \to 0} \frac{f'(x)}{g'(x)}$.

Sol. We know that f and g are not differentiable at the origin (x = 0).

Now $\quad \lim_{x \to 0} \frac{f(x)}{g(x)} = \lim_{x \to 0} \frac{|x|}{3|x|} = \frac{1}{3}$.

We have $\quad f(x) = x$, if $x \geq 0$ and $f(x) = -x$, if $x < 0$,

and $\quad g(x) = 3x$, if $x \geq 0$ and $g(x) = -3x$, if $x < 0$.

Clearly, $f'(x) = \begin{cases} 1, \text{ if } x > 0 \\ -1, \text{ if } x < 0 \end{cases}, g'(x) = \begin{cases} 3, \text{ if } x > 0 \\ -3, \text{ if } x < 0. \end{cases}$

Now $\lim\limits_{x \to 0-} \frac{f'(x)}{g'(x)} = \lim\limits_{x \to 0} \frac{-1}{-3} = \frac{1}{3},$

$\lim\limits_{x \to 0-} \frac{f'(x)}{g'(x)} = \lim\limits_{x \to 0} \frac{1}{3} = \frac{1}{3}.$

Hence $\lim\limits_{x \to 0} \frac{f'(x)}{g'(x)} = \frac{1}{3} = \lim\limits_{x \to 0} \frac{f(x)}{g(x)}.$

Ex : Let f and g be two functions with domain D satisfying $g(x) = x f(x) \ \forall \ x \in D$. Show that if f be continuous at $x = 0 \in D$, then g is differentiable at $x = 0$.

Sol. We have

$$\lim_{x \to 0} \frac{g(x) - g(0)}{x - 0} = \lim_{x \to 0} \frac{xf(x) - 0}{x} = \lim_{x \to 0} f(x) = f(0),$$

since f is continuous at $x = 0$.

Hence $g'(0) = f(0)$, which shows that g is differentiable at $x = 0$.

Sign of the Derivative

Suppose a function f is defined in [a, b] and f' (c) exists for any point $c \in [a, b]$.

Let $f'(c) > 0$.

Since $\lim\limits_{x \to c} \frac{f(x) - f(c)}{x - c} = f'(c)$, so for $\varepsilon > 0$, $\exists$ some $\delta > 0$, such that

$$\left| \frac{f(x) - f(c)}{x - c} - f'(c) \right| < \varepsilon, \text{ when } |x - c| < \delta.$$

$$\Rightarrow \left. \begin{array}{l} f'(c) - \varepsilon < \dfrac{f(x) - f(c)}{x - c} < f'(c) + \varepsilon, \\ \text{when } x \in \,]\, c - \delta, c + \delta \,[, \ x \neq c \end{array} \right\} \quad ...(1)$$

Let us choose $\varepsilon > 0$ so that $\varepsilon < f'(c) \Rightarrow f'(c) > \varepsilon \ \Rightarrow \ f'(c) - \varepsilon > 0$ ...(2)

From (1), we consider $f'(c) - \varepsilon < \frac{f(x) - f(c)}{x - c}$

$$\Rightarrow \quad \frac{f(x) - f(c)}{x - c} > f'(c) - \varepsilon > 0, \text{ using (2)}$$

$$\Rightarrow \quad \frac{f(x) - f(c)}{x - c} > 0, \text{ when } x \in (c - \delta, c + \delta), \ x \neq c.$$

$\Rightarrow \quad f(x) - f(c) > 0$, when $x \in (c, c + \delta)$ $\quad (\because \ x > c)$

and $\quad f(x) - f(c) < 0$, when $x \in (c - \delta, c)$. $\quad (\because \ x < c)$

$$\therefore \quad \begin{cases} f(x) > f(c), \text{ when } x \in \,]\, c, c + \delta \,[, \\ f(x) < f(c), \text{ when } x \in \,]\, c - \delta, c \,[. \end{cases}$$

Hence $\left.\begin{array}{l} f'(c) > 0 \Rightarrow \exists \text{ some } \delta > 0 \text{ such that} \\ \quad f(x) > f(c), \forall\ x \in]\,c, c+\delta\,[\\ \quad f(x) < f(c), \forall\ x \in]\,c-\delta, c\,[. \end{array}\right\}$ ...(A)

This property is often stated as follows :

If f'(c) > 0, then f is increasing at c.

Let f' (c) < 0.

We define a function g as follows :

$$g(x) = -f(x) \quad \forall \quad x \in [a, b].$$

Then $g'(c) = -f'(c) > 0$.

Now $g'(c) > 0 \quad \Rightarrow \quad \exists$ some $\delta > 0$ such that

$g(x) > g(c) \quad \Rightarrow \quad f(x) < f(c)\ \forall\ x \in (c, c+\delta)$,

$g(x) < g(c) \quad \Rightarrow \quad f(x) > f(c)\ \forall\ x \in (c-\delta, c)$.

Hence $\left.\begin{array}{l} \text{If } f'(c) < 0 \Rightarrow \exists \text{ some } \delta > 0 \text{ such that} \\ \quad f(x) < f(c), \forall\ x \in]\,c, c+\delta\,[, \\ \quad f(x) > f(c), \forall\ x \in]\,c-\delta, c\,[, \end{array}\right\}$ ...(B)

In other words,

If f' (c) < 0, then f is decreasing at c.

We now consider the end points. It may be show that

$$\left.\begin{array}{l} f'(a) > 0 \Rightarrow f(x) > f(a), \forall\ x \in]\,a, a+\delta\,[, \\ f'(a) < 0 \Rightarrow f(x) < f(a), \forall\ x \in]\,a, a+\delta\,[, \end{array}\right\} \quad \text{...(C)}$$

$$\left.\begin{array}{l} f'(b) > 0 \Rightarrow f(x) < f(b), \forall\ x \in]\,b-\delta, b\,[, \\ f'(b) < 0 \Rightarrow f(x) > f(b), \forall\ x \in]\,b-\delta, b\,[. \end{array}\right\} \quad \text{...(D)}$$

Note : The students are strongly advised to remember the results (A), (B), (C) and (D) as these will be frequently used.

Darboux's Theorem

Theorem : If a function f defined in [a, b] is such that

(i) f is differentiable in [a, b]

(ii) f' (a) and f' (b) are of opposite signs,

then there exists some point $c \in (a, b)$ such that $f'(c) = 0$.

Proof. Let us suppose that $f'(a) > 0$ and $f'(b) < 0$.

Since $f'(a) > 0 \Rightarrow \exists$ some $\delta_1 > 0$ such that

$f(x) > f(a),\ \forall\ x \in (a, a+\delta_1)$. ...(1)

Since $f'(b) < 0 \Rightarrow \exists$ some $\delta_2 > 0$ such that

$f(x) > f(b),\ \forall\ x \in (b-\delta_2, b)$. ...(2)

Since f is differentiable in [a, b], f is continuous in [a, b] and hence bounded there in and the bounds are attained.

Let $K = \sup f$ and $k = \inf f$. Then $\exists$ points c, d $\in$ [a, b] such that

$f(c) = K$ and $f(d) = k$. ...(3)

Now $K = \sup f \Rightarrow f(x) \le K,\ \forall\ x \in [a, b]$. ...(4)

We shall show that $c \neq a$ and $c \neq b$.

Let, if possible, $c = a$. Then $f(c) = f(a)$ or $K = f(a)$, using (3)

$\Rightarrow \quad f(x) > K, \ \forall \ x \in (a, a + \delta_1)$, by (1)

This contradicts (4). Thus $c \neq a$.

Similarly, we can show that $c \neq b$. Thus $c \in (a, b)$.

Finally, we show that $f'(c) = 0$.

Case I. Let $f'(c) > 0$.

$\Rightarrow \quad \exists$ some $\delta_3 > 0$ such that $f(x) > f(c), \ \forall \ x \in (c, c + \delta_3)$

$\Rightarrow \quad f(x) > K, \ \forall \ x \in (c, c + \delta_3)$, by (3)

This contradicts (4) and so $f'(c) \geq 0$.

Case II. Let $f'(c) < 0$.

$\Rightarrow \quad \exists$ some $\delta_4 > 0$ such that $f(x) > f(c), \ \forall \ x \in (c - \delta_3, c)$

$\Rightarrow \quad f(x) > K, \ \forall \ x \in (c - \delta_4, c)$, by (3)

This contradicts (4) and so $f'(c) \leq 0$. Hence $f'(c) = 0$.

Remark. Darboux's Theorem may be restated as follows :

Let f be defined and differentiable on $[a, b]$. If $f'(a) f'(b) < 0$, then there exists some point $c \in (a, b)$ such that $f'(c) = 0$.

Proof. It is clear that

$f'(a) f'(b) < 0 \Rightarrow f'(a)$ and $f'(b)$ are of opposite signs.

Now proceed like Theorem.

Theorem. If f is differentiable in $[a, b]$ and $f'(a) \neq f'(b)$, then for each number **k** lying between $f'(a)$ and $f'(b)$, $\exists$ some point $c \in (a, b)$ such that

$f'(c) = k$.

Proof. We are given that $f'(a) < k < f'(b)$. ...(1)

Let $\quad g(x) = f(x) - kx \ \ \forall \ x \in [a, b]$. ...(2)

Since f is differentiable in [a,b] and kx is also differentiable in [a,b], therefore, by (2), g is differentiable in [a,b].

Now $\quad g'(x) = f'(x) - k$ ...(3)

$\Rightarrow \quad g'(a) = f'(a) - k < 0$ and $g'(b) = f'(b) - k > 0$, by (1).

Thus g is differentiable in $[a, b]$ and $g'(a)$, $g'(b)$ are of opposite signs. So by Darboux's Theorem, $\exists$ some point $c \in (a, b)$ such that

$$g'(c) = 0$$

$\Rightarrow \quad f'(c) - k = 0$, by (3)

Hence $\quad f'(c) = k$.

Ex : If f be differentiable at a point c, then show that $|f|$ is also differentiable at c, provided $f(c) \neq 0$. Show by means of an example that if $f(c) = 0$, then f may be differentiable at c and $|f|$ may not be differentiable at c.

Sol. Since f is differentiable at c, f is continuous at c.

Now $\quad f(c) \neq 0 \Rightarrow f(c) > 0$ or $f(c) < 0$.

Consequently, $\exists \ \delta_1 > 0$ and $\delta_2 > 0$ such that

$$f(x) > 0 \quad \forall \quad x \in (c - \delta_1, c + \delta_1)$$

or $$f(x) < 0 \quad \forall \quad x \in (c - \delta_2, c + \delta_2).$$

Thus $\quad |f(x)| = f(x) \quad \forall \quad x \in (c - \delta_1, c + \delta_1)$

or $\quad | f(x) | = - f(x) \quad \forall \quad x \in (c - \delta_2, c + \delta_2)$.

In either case, it follows that | f | is differentiable at c, since f is differentiable at c.

However, | f | may not be differentiable at c, if (c) = 0.

For example let $f(c) = x \; \forall \; x \in \mathbb{R}$. Then $f(0) = 0$ and

$$| f(x) | = | x | \quad \forall \quad x \in \mathbb{R}.$$

Clearly, | f | is not differentiable at x = 0, although f is differentiable at x = 0.

Ex : If f be defined and differentiable on [a, b], f(a) = f(b) = 0, and f'(a) and f'(b) are of the same sign, then prove that f must vanish at least once in (a, b).

Sol. We are given that f(a) = f(b) = 0.

Since f'(a) and f'(b) are of the same sign, we may take

$$f'(a) > 0 \quad \text{and} \quad f'(b) > 0.$$

Now $\quad f'(a) > 0 \Rightarrow \exists$ some $\delta_1 > 0$ such that

$$f(x) > f(a) = 0 \quad \forall \quad x \in (a, a + \delta_1].$$

Again $\quad f'(b) > 0 \Rightarrow \exists$ some $\delta_2 > 0$ such that

$$f(x) > f(b) = 0 \quad \forall \quad x \in [b - \delta_2, b).$$

Thus $\quad f(a + \delta_1) > 0$ and $f(b - \delta_2) < 0$. ...(1)

Obviously, $\quad [a + \delta_1, b - \delta_2] \subset [a, b]$. ...(2)

Since f is differentiable in [a, b], f is continuous in [a, b] and consequently by (2), f is continuous in $[a + \delta_1, b - \delta_2]$, where $f(a + \delta_1)$ and $f(b - \delta_2$ are of opposite signs, by (1). Hence by Theorem , there exists some $c \in (a + \delta_1, b - \delta_2) \subset (a, b)$ such that f(c) = 0. Hence f must vanish at least once in (a, b).

Ex : If f is differentiable on [a, b], f(a) = f(b) = 0, and $f(x) \neq 0$ for any x in(a, b), then prove that f'(a) and f'(b) must be of opposite signs.

Sol. Let, if possible, f'(a) and f'(b) be of the same sign. Also f(a) = f(b) = 0. By Example there exists some $c \in (a, b)$ such that f(c) = 0. This contradicts the given hypothesis that $f(x) \neq 0$ for all x in (a, b). Hence f'(a) and f'(b) must be of opposite signs.

Ex : Let $f : [-1, 1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 0, & -1 \le x \le 0 \\ 1, & 0 < x \le 1 \end{cases}$$

Does there exists a function g such that $g'(x) = f(x) \; \forall \; x \in [-1, 1]$?

Sol. Let, if possible, there exist a function $g : [-1, 1] \to \mathbb{R}$ such that

$$g'(x) = f(x) = \begin{cases} 0, & -1 \le x \le 0 \\ 1, & 0 < x \le 1. \end{cases}$$

It follows that g is differentiable on [– 1, 1] and $g'(-1) \neq g'(1)$

$[\because \; g'(-1) = 0$ and $g'(1) = 1]$.

By Theorem g' must assume every value between 0 and 1.

$[\because \; g'(-1) = 0$ and $g'(1) = 1.]$

But this is impossible, since

$$g'(x) = 1 \quad \forall \quad x \in (0, 1).$$

Hence there does not exist any function g such that

$$g'(x) = f(x) \; \forall \; x \in [-1, 1].$$

7. INTERMEDIATE VALUE PROPERTY

A real function is said to have the **Intermediate value property** on an interval [a, b] if, for each value v between f(a) and f(b), there is some $c \in (a, b)$ such that $f(c) = v$. Thus, a function with the intermediate value property takes all intermediate values between any two points.

The simplest, and most important, examples of functions with this property are the continuous functions.

Theorem by (Intermediate Value Theorem)

If a function f is continuous in [a, b] and $f(a) \neq f(b)$, then f assumes every value between f(a) and f(b).

Proof. Suppose that $f(a) < f(b)$ and let M be any number lying between f(a) and f(b), i.e.,

$$f(a) < M < f(b) \qquad ...(1)$$

We have to show that $\exists$ some point $c \in [a, b]$ such that $f(c) = M$.

Let us define a function ϕ on [a, b] as follows :

$$\phi(x) = f(x) - M, \quad \forall \quad x \in [a, b] \qquad ...(2)$$

Since f is given to be continuous in [a, b] and M being a constant is always continuous, so ϕ is continuous in [a, b].

From (2), $\phi(a) = f(a) - M$ and $\phi(b) = f(b) - M$.

By (1), $f(a) - M < 0$ and $f(b) - M > 0$

$\Rightarrow$ $\phi(a)$ and $\phi(b)$ are of opposite signs,

By Theorem there exists some point $c \in (a, b)$ such that

$$\phi(c) = 0 \quad \Rightarrow \quad f(c) - M = 0, \text{ by (2)}$$

Hence $f(c) = M$, $c \in (a, b)$.

Remark. If a function f is not continuous on a closed interval, then the conclusion of the Intermediate Value Theorem may not hold:

Consider a function on [0, 1] as follows :

$$f(x) = x + 1 \quad \forall \quad x \in (0, 1].$$

$$f(0) = 0.$$

Obviously, f is continuous on (0, 1]. Further

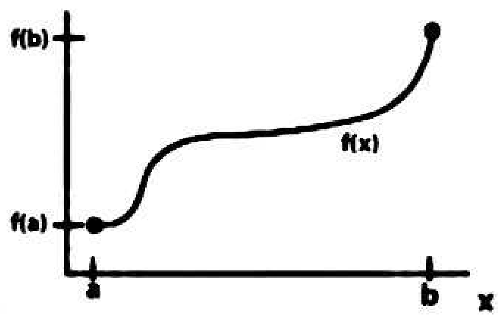
$$\lim_{x \to 0-} f(x) = \lim_{x \to 0} (x + 1) = 1. \text{ Also } f(0) = 0.$$

$\therefore$ $\lim_{x \to 0+} f(x) \neq f(0)$ and so f is not continuous at 0.

i.e., f is not continuous in the closed interval [0, 1].

Now $f(0) = 0$, $f(1) = 2$.

We see that $f(0) < 1 < f(1)$. But there does not exist y any $x \in (0, 1]$ such that $f(x) = 1$.



Ex. Is there a solution to $x^5 - 2x^3 - 2 = 0$, where $x \in [0, 2]$?

At $x = 0$, we have $0^5 - 2 \times 0^3 - 2 = -2$.

At $x = 2$, we have $2^5 - 2 \times 2^3 - 2 = 14$.

So that IVT implies that there is a solution to $x^5 - 2x^3 - 2 = 0$ in the interval [0, 2].

Ex. Suppose that f is continuous on [0, 1] and f(0) = f(1). Let n be any positive integer, then prove that there is some number x such that

$$f(x) = f\left(x + \frac{1}{n}\right).$$

Define $g(x) = f(x) - f\left(x + \frac{1}{n}\right)$.

Consider the set of numbers $S = \left\{f(0), f\left(\frac{1}{n}\right), f\left(\frac{2}{n}\right), \ldots, f(1)\right\}$

Let k be such that $f\left(\frac{k}{n}\right)$ is the largest number in S. Suppose that $k \neq 0$ and $k \neq n$.

Then $g\left(\frac{k}{n}\right) = f\left(\frac{k}{n}\right) - f\left(\frac{k+1}{n}\right) \geq 0$, and $g\left(\frac{k-1}{n}\right) = f\left(\frac{k-1}{n}\right) - f\left(\frac{k}{n}\right) \leq 0$.

By the Intermediate value theorem, there is $c \in \left[\frac{k-1}{n}, \frac{k}{n}\right]$ with g(c) = 0, so that $f(c) - f\left(c + \frac{1}{n}\right)$ = 0, or $f(c) = f\left(c + \frac{1}{n}\right)$ as desired.

Finally, if the largest number in S is f(0) = f(1), then the same argument works with k chosen such that $f\left(\frac{k}{n}\right)$ is the minimum number in S.

Note that if f(0) is both the largest and smallest number in S, then they are all the same and $f(0) = f\left(\frac{1}{n}\right)$.

Suppose that f is continuous on [0, 1] and f(0) = f(1). Let ε be the hyperreal unit, then prove that there is some number x such that

$$f(x) = f(x + \varepsilon).$$

First, assume f(x) is not constant on [0, 1]. The result holds trivially if it is.

Then, let $g(x) = f(x) - f(x + \varepsilon)$.

Since f is not constant, there exists a $c \in [0, 1]$ such that f(c) is a maximum (or a minimum, but assume for now that it is a max, a min is handled similarly).

Then $g(c) = f(c) - f(c + \varepsilon) \geq 0$ and $g(c - \varepsilon) = f(c - \varepsilon) - f(c) \leq 0$

Now, by the Intermediate Value Theorem, if $\varepsilon > 0$, there exists a $y \in [c - \varepsilon, c]$ such that $g(y) = 0$.

(If $\varepsilon < 0$, $y \in [c, c - \varepsilon]$, instead.)

Thus, $f(y) = f(y + \varepsilon)$, as wanted.

If $\varepsilon = 0$, $f(x) = f(x + \varepsilon)\ \forall x$

Ex. Apply intermediate value property to show that the equation $x^5 - 3x^2 = -1$ has a solution in the interval [0, 1].

Sol. Let $f(x) = x^5 - 3x^2$. Then f(x) is a continuous function with f(0) = 0 and f(1) = –2. As –1 is a value between –2 and 0, the intermediate value property of continuous functions indicates that f(x) = –1 must have a solution in the interval [0, 1].

Ex. Apply intermediate value property to show that the equation $x^5 - 5x^3 + 3 = 0$ has a solution in the interval [–1, 1].

Sol. Let $f(x) = x^5 - 5x^3 + 3$. Then f(x) is a continuous function with f(–1) = 7 and f(1) = –1. As 0 is a value between –1 and 7, the intermediate value property of continuous functions indicates that f(x) = 0 must have a solution in the interval [–1, 1].

Ex. Apply intermediate value property to show that the equation $\sqrt{x^6 + 5x^4 + 9} = 3.5$ has a solution in the interval [0, 1].

Sol. Let $f(x) = \sqrt{x^6 + 5x^4 + 9}$. Then f(x) is a continuous function with f(0) = 3 and f(1) = 4. As 3.5 is a value between 3 and 4, the intermediate value property of continuous functions indicates that f(x) = 3.5 must have a solution is the interval [0, 1].

Ex. Show that the equation (1 – x)cosx = sinx has at least one solution in (0, 1).

Sol. Set f(x) = (1 – x)cosx – sinx. Then f(0) = 1 and f(1) = –sin1 < 0. By the intermediate value property there is $x_0 \in (0, 1)$ such that $f(x_0) = 0$.

Ex. Let f : [0, 1] → [0, 1] be continuous. Show that f has a fixed point in [0, 1]; that is, there exists $x_0 \in [0, 1]$ such that $f(x_0) = x_0$.

Sol. Define the function g(x) = f(x) – x on [0, 1]. Then f is continuous, g(0) ≥ 0 and g(1) ≤ 0. Use the intermediate value property (IVP).

Ex. Let f : [a, b] → $\mathbb{R}$ be a continuous function. Show that the range {f(x) : ∈ [a, b]} is a closed and bounded interval.

Sol. Since f is a continuous function, there exist x_0, $y_0 \in [a, b]$ such that $f(x_0) = m = \inf f$ and $f(y_0) = M = \sup f$. Suppose $x_0 < y_0$. By the IVP, for every $\alpha \in [m, M]$ there exists $x \in [x_0, y_0]$ such that $f(x) = \alpha$. Hence f[(a, b]) = [m, M].

Ex. Show that a polynomial of odd degree has at least one real root.

Sol. Let $p(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$, $a_n \neq 0$ and n be odd. Then $p(x) = x^n \left(a_n + \frac{a_{n-1}}{x} + \ldots + \frac{a_1}{x^{n-1}} + \frac{a_0}{x^n}\right)$. If $a_n > 0$, then $p(x) \to \infty$ as $x \to \infty$ and $p(x) \to -\infty$ as $x \to -\infty$. Thus by the IVP, there exists x_0 such that $p(x_0) = 0$. Similar argument for $a_n < 0$.

Unit-5 ‖ **Functions of One Variable-II**

1. ROLLE'S THEOREM : STATEMENT

If a function f defined on [a, b] is such that it is :

[R_1] continuous in the closed interval [a, b]

[R_2] differentiable in the open interval (a, b)

and [R_3] f(a) = f(b),

then there exists atleast one point $c \in (a, b)$ such that $f'(c) = 0$.

Proof : Since f is continuous in the closed interval [a, b], therefore by the property of continuity f will be bounded and will attain bounds in [a, b].

Let m and M be the infimum and supremum of f in [a, b], then there exist c and d in [a, b] such that f(c) = M and f(d) = m

Case 1 : When M = m, then

$$f(x) = M = m, \quad \forall\ x \in [a, b]$$

$$\Rightarrow \quad f'(x) = 0, \quad \forall\ x \in [a, b]$$

$$\Rightarrow \quad f'(c) = 0, \text{ where } c \in (a,b)$$

Thus the theorem hold good at any point c of (a, b).

Case 2 : When $M \neq m$, then by R_3, f(a) = f(b), and atleast one of the numbers M and m be different from f(a) and f(b) i.e.,

$$M = f(c) \neq f(a), f(b)$$

$$\Rightarrow \quad c \neq a, b \Rightarrow c \in (a, b)$$

Since the function f is differentiable in (a, b), then the derivative exist at a point c.

Therefore RHD and LHD are

$$\lim_{x \to c+0} \frac{f(x)-f(c)}{x-c} \text{ and } \lim_{x \to c-0} \frac{f(x)-f(c)}{x-c}$$

both exist and each equal to f'(c).

Since f(c) is the maximum value of f(x) in (a,b),

$$\therefore \quad f(c + h) < f(c) \text{ and } f(c - h) < f(c) \qquad ...(1)$$

Now $\quad \text{RHD} = \lim_{x \to c+0} \frac{f(x)-f(c)}{x-c}$

$= \lim_{h \to 0} \frac{f(c+h)-f(c)}{h}$ [Putting x = c + h]

≤ 0 [$\because$ by(1) f(c + h) – f(c) < 0]

and $\quad \text{LHD} = \lim_{x \to c-0} \frac{f(x)-f(c)}{x-c}$

$= \lim_{h \to 0} \frac{f(c-h)-f(c)}{-h}$ [Putting x = c – h]

≥ 0 [$\because$ by(1) f(c – h) – f(c) < 0]

Thus both the derivatives are of opposite signs.

Therefore both these derivatives will be equal only when $f'(c) = 0$.

Geometrical Interpretation of Rolle's Theorem

Statement : It a curve has a tangent at every point there of and the ordinates of its extremities A and B are equal i.e. $f(a) = f(b)$; then there exists atleast one point P of the curve other than A and B, the tangent at which is parallel to x-axis.

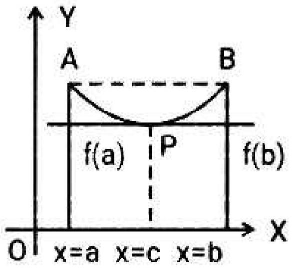
Case 1 : When $f(x) = k$ (constant),

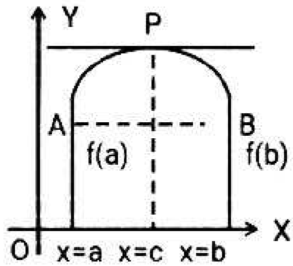
Then the graph of this function will be a straight line parallel to the x-axis whose equation is $y = k$ and the derivative

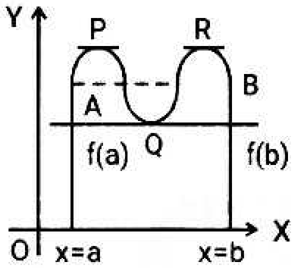
$$f'(x) = 0, \qquad \forall\ x \in [a, b]$$

Case 2 : When $f(x) \neq k$ (constant),

Then the graph of this function is a continuous curve of the following type in [a, b] as shown below:





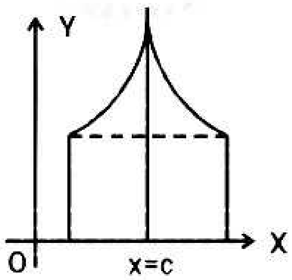


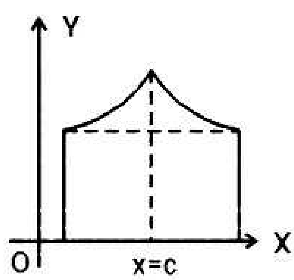
From these figures it is quite intuitively that there must be atleast one point $P(x = c)$ (may be more) of the curve other than A and B, the tangent at which is parallel to x-axis and at these points $f'(c) = 0$.

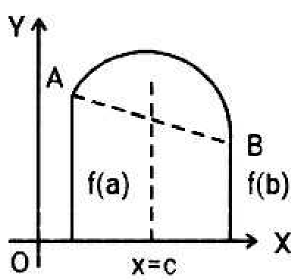
Case of Failure : The theorem will not hold good if any of the three conditions fails to hold good i.e.,

1. the function f is discontinuous at $x = c$.
2. the derivative does not exist at $x = c$.
3. $f(a) \neq f(b)$.

The following figures will illustrate these cases of failure:







Algebraic Interpretation of Rolle's theorem:

Statement : If $f(x)$ be a polynomial in x and $x = a$ and $x = b$ are any two roots of the equation $f(x) = 0$, then there exists at least one root of the equation $f'(x) = 0$, which lies between a and b.

Remarks : The Rolle's theorem is not applicable for those functions which do not satisfy even one condition of Roll's theorem.

Another useful form of Rolle's Theorem:

If a function f defined on [a, a + h] is such that it is:

[R_1] continuous in [a, a + h],

[R_2] differentiable in (a, a + h) and

[R_3] f(a) = f(a + h),

then $\exists\ \theta \in (0, 1)$ such that $f'(a + \theta h) = 0$.

Ex. Verify Rolle's theorem for the function in the interval mentioned against:

$$f(x) = x(x + 3)e^{-x/2},\ x \in [-3, 0]$$

Sol. R_1. Since the given function x(x + 3), x is a polynomial of x, therefore it is continuous for every value of x and $e^{-x/2}$ is also continuous for every value of x.

Therefore their product is also continuous for every value of x

Hence the function is continuous, particularly in [–3,0].

R_2. $f'(x) = (2x + 3)\, e^{-x/2} + x(x + 3) \cdot e^{-x/2}\left(-\frac{1}{2}\right) = \frac{1}{2}e^{-x/2}\,(6 + x - x^2)$

which is not infinite or indeterminate at any point of (–3,0).

Thus f(x) is differentiable in (–3,0).

R_3. f(–3) = 0 = f(0)

Thus the given function f satisfies all the three conditions of Roll's theorem.

Hence Verified.

Therefore there must be atleast one point c in (–3,0), where

$$f'(c) = 0 \Rightarrow \frac{1}{2}e^{-c/2}\,(6 + c + c^2) = 0$$

Since $e^{-c/2}$ is not zero for any finite value c,

therefore $6 + c - c^2 = 0 \Rightarrow c = -2, 3$

One value of $c = -2 \in (-3,0)$ **Hence Verified.**

Ex. Verify Rolle's Theorem for

$$f(x) = \sqrt{1 - x^2} \text{ in } [-1, 1].$$

Sol. The given function being an algebraic function of x is continuous in [– 1, 1].

Now $f'(x) = \frac{-x}{\sqrt{1 - x^2}}$, for $-1 < x < 1$.

So f(x) is derivable in] – 1, 1[. Also f(1) = f(–1) = 0.

Thus f(x) satisfies all the conditions of Rolle's Theorem.

So there exists a point $c \in\ ]-1, 1[$ such that f′(c) = 0

Clearly $f'(c) = 0$ for $c = 0 \in\ ]-1, 1[$.

Hence the given function satisfies the hypothesis and the conclusion of Rolle's Theorem.

Ex. Examine the validity of the hypotheses and the conclusion of Rolle's theorem for the function

$$f(x) = 1 - (x - 1)^{2/3} \text{ on } [0, 2].$$

Sol. The given function, being an algebraic function of x is continuous in [0, 2].

Now $f'(x) = -\frac{2}{3}(x - 1)^{-1/3} = \frac{-2}{3(x-1)^{1/3}}$

$\Rightarrow$ f′(x) does not exist at x = 1 $\Rightarrow$ f is not derivable at x = 1

$\Rightarrow$ f is not derivable in]0, 2[.

But f(0) = f(2) = 0.

Hence f does not satisfy all the conditions of Rolle's theorem. Clearly, the conclusion of Rolle's theorem is not valid for the given function.

Ex. Show that between any two roots of $e^x \cos x = 1$, there exists at least one root of $e^x \sin x - 1 = 0$.

Sol. Let α and β be any two distinct roots of $e^x \cos x = 1$. ...(1)

$\therefore$ $e^\alpha \cos \alpha = 1$ and $e^\beta \cos \beta = 1$.

Define a function f as follows :

$$f(x) = e^{-x} - \cos x, \ \forall \ x \in [\alpha, \beta]. \quad ...(2)$$

Obviously f is continuous in $[\alpha, \beta]$ and f is derivable in $]\alpha, \beta[$.

Indeed, $f'(x) = -e^{-x} + \sin x \ \forall \ x \in]\alpha, \beta[$.

From (2), $f(\alpha) = e^{-\alpha} - \cos \alpha = \dfrac{1 - e^\alpha \cos\alpha}{e^\alpha} = 0$, by (1)

Similarly, $f(\beta) = 0$ and so $f(\alpha) = f(\beta)$.

Thus f satisfies all the conditions of Rolle's Theorem in $[\alpha, \beta]$ and so there exists some $\gamma \in]\alpha, \beta[$ such that $f(\gamma) = 0$.

$\Rightarrow$ $\sin \gamma - e^{-\gamma} = 0 \Rightarrow e^\gamma \sin \gamma - 1 = 0, \ \alpha < \gamma < \beta$.

Hence γ is a root of $e^x \sin x - 1 = 0$, $\alpha < \gamma < \beta$.

Ex. Prove that between any two real roots of $e^x \sin x = x$, there is at least one root of $\cos x + \sin x = e^{-x}$.

Sol. Let $x = \alpha$, $x = \beta$ be any two distinct roots of $e^x \sin x = x$.

$\therefore$ $e^\alpha \sin \alpha = \alpha$ and $e^\beta \sin \beta = \beta$. ...(1)

Define a function f on $[\alpha, \beta]$ as follows :

$$f(x) = e^x \sin x - x \ \forall \ x \in [\alpha, \beta]. \quad ...(2)$$

Clearly, (i) f continuous in $[\alpha, \beta]$, (ii) f is derivable in $]\alpha, \beta[$ and $f(\alpha) = 0 = f(\beta)$,. by (1). Hence by Rolle's theorem, there exists some $\gamma \in]\alpha, \beta[$ i.e., $\alpha < \gamma < \beta$ such that $f'(\gamma) = 0$. ...(3)

From (2), $f'(x) = e^x \sin x + e^x \cos x - 1$

$\Rightarrow$ $f'(g) = e^\gamma (\sin \gamma + \cos \gamma) - 1$

$\therefore$ $e^\gamma (\sin \gamma + \cos \gamma) - 1 = 0$, using (3)

or $\sin \gamma + \cos \gamma = 1/e^\gamma = e^{-\gamma}$, $\alpha < \gamma < \beta$.

Hence γ is a root of $\sin x + \cos x = e^{-x}$, where $\alpha < \gamma < \beta$.

Ex. If p(x) is polynomial and $k \in R$, prove that between any two real roots of $p(x) = 0$, there is a roots of $p'(x) + k\,p(x) = 0$.

Sol. Let $f(x) = e^{kx} p(x), \ x \in R$. ...(1)

Let α and β be any two real roots of p(x), where $\alpha < \beta$. Then $p(\alpha) = 0 = p(\beta) \Rightarrow f(\alpha) = 0 = f(\beta)$, by (1).

Obviously, (i) f is continuous in $[\alpha, \beta]$, (ii) f is derivable in $]\alpha, \beta[$ and (iii) $f(\alpha) = f(\beta)$. Hence, by Rolle's theorem, there exists some $\gamma \in]\alpha, \beta[$ such that $f'(\gamma) = 0$.

From (1), $f'(x) = k\,e^{kx} p(x) + e^{kx} p'(x) = e^{kx} [p'(x) + kp(x)]$.

$\therefore$ $0 = f'(\gamma) = e^{k\gamma} [p'(\gamma) + k\,p(\gamma)]$

$\Rightarrow$ $p'(\gamma) + k\,p(\gamma) = 0$, as $e^{k\gamma} \neq 0$.

Hence γ is a root of $p'(x) + kp(x) = 0$, where $\alpha < \gamma < \beta$.

2. MEAN VALUE THEOREM

Lagrange's Mean Value Theorem

Statement : If a function f with domain [a, b] is such that it is

[L_1] continuous in [a, b], and

[L_2] differentiable in (a, b),

then $\exists$ c $\in$ (a, b) such that:

$$f'(c) = \frac{f(b)-f(a)}{b-a}.$$

Proof : Let us define a new function ϕ with domain [a, b] involving the given function f(x) as follows :

$$\phi(x) = f(x) + Ax \qquad ...(1)$$

where A is a constant to be determined such that $\phi(a) = \phi(b)$ ...(2)

$\Rightarrow \quad f(a) + Aa = f(b) + Ab$

$$\Rightarrow \quad A = -\frac{f(b)-f(a)}{b-a} \qquad ...(3)$$

We observe that the function ϕ is also defined on [a, b] and is the sum of two continuous functions on [a, b] and differentiable in (a, b), therefore ϕ is :

R_1. ϕ,[a, b] is continuous in [a, b]

R_2. ϕ,(a, b) is differentiable in (a, b)

and **R_3.** $\phi(a) = \phi(b)$ [by the condition of constant]

Thus ϕ satisfies all the three conditions of Rolle's theorem, therefore accordingly, there must be atleast one point c in (a, b) such that

$\phi'(c) = 0$

$\Rightarrow \quad f'(c) + A = 0 \Rightarrow f'(c) = -A$

$$\Rightarrow \quad f'(c) = \frac{f(b)-f(a)}{b-a}, \qquad a < c < b \qquad \text{[by (3)]}$$

Important Particular Case : If f(a) = f(b), then it reduces to Rolle's theorem.

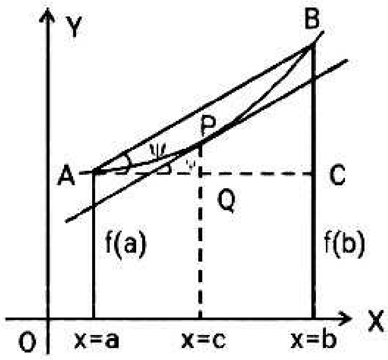
Geometrical Interpretation of Lagrange's MV Theorem:

If a curve y = f(x) is continuous between two given points whose abscissae are x = a and x = b respectively and a tangent can be drawn to the curve at every point, then there exists atleast one point x = c, c $\in$ (a, b) such that the tangent there at is parallel to the chord joining the two given end points.

Let the arc APB represents the graph of the function y = f(x) and a and b be the ordinates of A and B respectively. Join AB. Draw perpendiculars from A and B on x-axis. Let the chord AB makes an angle Ψ with the x-axis, then from the right angled triangle ACB,

$$\tan \Psi = \frac{CB}{AC}$$

$$= \frac{f(b)-f(a)}{b-a} = f'(c), \; a < c < b$$



Another Useful Form

Statement : A function f defined on [a, a + h] is such that it is :

(1) Continuous in [a, a + h];

(2) Differentiable in (a, a + h)

then $\exists\ \theta \in (0, 1)$ such that :

$$f(a + h) - f(a) = h\, f'(a + \theta h).$$

If we put $b - a = h$ or $b = a + h$ in Lagrange's MV theorem, then any point c can be taken as $c = a + \theta h$ between a and b, where $0 < \theta < 1$.

Therefore by Lagrange MVT, we have

$$f'(a + \theta h) = \frac{f(a+h) - f(a)}{h}$$

$\Rightarrow \quad f(a + h) - f(a) = h\, f'(a + \theta h),\ 0 < \theta < 1.$

Important Deduction from Lagrange's MV Theorem :

Theorem : If a function f is

(1) continuous in [a, b]

(2) differentiable in (a, b), $\forall\ x \in (a, b)$, then

(a) $f'(x) = 0 \Rightarrow f$ is a constant function in (a, b).

(b) $f'(x) > 0 \Rightarrow f$ is strictly increasing in [a, b].

(c) $f'(x) < 0 \Rightarrow f$ is strictly decreasing in [a, b].

If the derivatives of two functions at every point of the interval (a, b) is same, then the functions differ by a constant.

Ex. Verify Lagrange's Mean Value Theorem for the function $f(x) = \sqrt{x^2 - 4}$ in [2, 4].

Sol. The function $f(x) = \sqrt{x^2 - 4}$ is an algebraic function of x, so f is continuous in [2, 4]. Also f is derivable in]2, 4[.

Now $f'(x) = \dfrac{x}{\sqrt{x^2 - 4}}\ \forall\ x \in\ ]2, 4[.$

Since f satisfies all the conditions of Lagrange's mean value theorem, therefore, there exists some $c \in\ ]2, 4[$ such that

$$\frac{f(4) - f(2)}{4 - 2} = f'(c)$$

$$\Rightarrow \quad 2\sqrt{3} - 0 = 2.\frac{c}{\sqrt{c^2 - 4}}$$

$\Rightarrow \quad \sqrt{3} = \dfrac{c}{\sqrt{c^2-4}} \Rightarrow 3(c^2 - 4) = c^2 \Rightarrow 2c^2 = 12 \Rightarrow c = \pm\sqrt{6}$.

Obviously, $c = \sqrt{6} \in]2, 4[$.

Hence the Lagrange's mean value theorem is verified.

Ex. Verify Lagrange's Mean Value Theorem for the function

$$f(x) = x(x-1)(x-2) \text{ in } \left[0, \frac{1}{2}\right]$$

Sol. Since f is a polynomial, f is continuous in $\left[0, \frac{1}{2}\right]$ and derivable in $]0, \frac{1}{2}[$. Thus there exists $c \in]0, \frac{1}{2}[$ such that

$$\frac{f\left(\frac{1}{2}\right) - f(0)}{\frac{1}{2} - 0} = f'(c). \qquad ...(1)$$

Now $f'(x) = (x - 1)(x - 2) + x(x - 1) + x(x - 2)$

or $f'(x) = 3x^2 - 6x + 2$. Now $f(0) = 0$, $f\left(\frac{1}{2}\right) = \frac{3}{8}$.

From (1), $\frac{3}{8} = \frac{1}{2}(3c^2 - 6c + 2)$

or $12c^2 - 24c + 5 = 0$ or $c = \dfrac{6 \pm \sqrt{21}}{6}$

Now $c = \dfrac{6+\sqrt{21}}{6} \notin]0, 1[$, $c = \dfrac{6-\sqrt{21}}{6} \in]0, 1[$.

Thus $c = \dfrac{6-\sqrt{21}}{6}$ and the theorem is verified.

Ex. Let f be defined and continuous in $[a - h, a + h]$ and derivable in $]a - h, a + h[$. Prove that there is a real number θ between 0 and 1 such that

$$f(a + h) - f(a - h) = h[f'(a + \theta h) + f'(a - \theta h)].$$

Sol. Define $\phi(x) = f(a + hx) - f(a - hx)$ on $[0, 1]$. ...(1)

$\therefore \quad \phi'(x) = hf'(a + hx) + hf'(a - hx)$. ...(2)

As x varies over $[0, 1]$, $a + hx$ varies over $[a, a + h]$ and $a - hx$ varies over $[a - h, a]$. Thus

(i) ϕ is continuous in $[0, 1]$.

(ii) ϕ is derivable in $]0, 1[$.

By Lagrange's mean value theorem, there exists θ, $0 < \theta < 1$, satisfying

$$\frac{\phi(1) - \phi(0)}{1 - 0} = \phi'(\theta) \text{ or } \phi(1) - \phi(0) = \phi'(\theta).$$

Hence $f(a + h) - f(a - h) = h[f'(a + \theta h) + f'(a - \theta h)]$, by (1) and (2).

Ex. Let f be defined and continuous on $[a - h, a + h]$ and derivable on $]a - h, a + h[$. Prove that there is real number θ between 0 and 1 for which

$$f(a + h) - 2f(a) + f(a - h) = h[f'(a + \theta h) - f'(a - \theta h)].$$

Sol. Consider $\phi(x) = f(a + hx) + f(a - hx)$ on $[0, 1]$.

Then (i) ϕ is continuous in $[0, 1]$, (ii) ϕ is derivable in $]0, 1[$. So by Lagrange's mean value theorem.

$$\frac{\phi(1)-\phi(0)}{1-0} = \phi'(\theta) \text{ for some } \theta \in]0, 1[$$

or $\quad \phi(1) - \phi(0) = \phi'(\theta)$

or $\quad [f(a + h) + f(a - h)] - [f(a) + f(a)] = h[f'(a + \theta h) - f'(a - \theta h)]$.

Hence $\quad f(a + h) - 2f(a) + f(a - h) = h[f'(a + \theta h) - f'(a - \theta h)]$.

Ex. Prove that if f be defined for all real x such that

$$| f(x) - f(y) | < (x - y)^2$$

for all real x and y, then f is constant.

Sol. Let $c \in R$. For $x \in c$, we have

$$\left|\frac{f(x)-f(c)}{x-c}\right| \le |x-c|, \text{ using (1)}.$$

If $\varepsilon > 0$, then on choosing $\delta = \varepsilon$, we obtain

$$\left|\frac{f(x)-f(c)}{x-c}\right| < \varepsilon, \text{ when } |x - c| < \delta.$$

$$\therefore \quad \lim_{x\to c} \frac{f(x)-f(c)}{x-c} = 0 \Rightarrow f'(c) = 0 \ \forall\ c \in R.$$

Thus $f'(x) = 0\ \forall\ x \in R$.

Hence f is a constant function.

Ex. Use mean value theorem to prove that

$$1 + x < e^x < 1 + xe^x, \quad \forall \quad x > 0.$$

Sol. We shall apply Lagrange's mean value theorem to the function $f(x) = e^x$ in the interval $[0, x]$. The function $f(x) = e^x$ is derivable in $[0, x]$ and, therefore, there exists some c, $0 < c < x$, such that

$$\frac{e^x - e^0}{x-0} = e^c \quad \text{or} \quad \frac{1}{x}(e^x - 1) = e^c. \qquad ...(1)$$

Since $0 < c < x$, so $1 = e^0 < e^c < e^x$. ...(2)

From (1) and (2), $1 < \frac{1}{x}(e^x - 1) < e^x$ or $x < e^x - 1 < xe^x$. $\quad (\because\ x > 0)$

Hence $1 + x < e^x < 1 + xe^x\ \forall\ x > 0$.

Ex. Show that

$$\frac{v-u}{1+v^2} < \tan^{-1} v - \tan^{-1} u < \frac{v-u}{1+u^2}, \text{ if } 0 < u < v,$$

and deduce that $\quad \dfrac{\pi}{4}+\dfrac{3}{25} < \tan^{-1}\dfrac{4}{3} < \dfrac{\pi}{4}+\dfrac{1}{6}$.

Sol. Applying Lagrange's mean value theorem to the function

$f(x) = \tan^{-1} x$ in [u, v], we obtain

$$\frac{f(v)-f(u)}{v-u} = f'(c) \text{ for some } c \in \,]\,u, v\,[$$

or $$\frac{\tan^{-1}v - \tan^{-1}u}{v-u} = \frac{1}{1+c^2}, \text{ for } u < c < v. \quad ...(1)$$

Now $c > u \Rightarrow 1 + c^2 > 1 + u^2 \Rightarrow \frac{1}{1+c^2} < \frac{1}{1+u^2}$. ...(2)

Again $c < v \Rightarrow 1 + c^2 < 1 + v^2 \Rightarrow \frac{1}{1+c^2} > \frac{1}{1+v^2}$. ...(3)

From (1), (2), (3) ; we obtain

$$\frac{1}{1+v^2} < \frac{\tan^{-1}v - \tan^{-1}u}{v-u} < \frac{1}{1+u^2}.$$

Hence $$\frac{v-u}{1+v^2} < \tan^{-1}v - \tan^{-1}u < \frac{v-u}{1+u^2}. \quad ...(4)$$

$(\because\ u < v \Rightarrow v - u > 0)$

Taking u = 1 and $v = \frac{4}{3}$ in (4), we obtain

$$\frac{3}{25} < \tan^{-1}\frac{4}{3} - \tan^{-1}1 < \frac{1}{6}.$$

or $$\frac{3}{25} < \tan^{-1}\frac{4}{3} - \frac{\pi}{4} < \frac{1}{6}.$$

Hence $\frac{\pi}{4} + \frac{3}{25} < \tan^{-1}\frac{4}{3} < \frac{\pi}{4} + \frac{1}{6}$.

Ex. Prove that $|\tan^{-1} x - \tan^{-1} y| \le |x - y|\ \forall\ x, y \in R$.

Sol. Let $f(t) = \tan^{-1} t$ in [x, y], where x < y.

By Lagrange's mean value theorem, there exists some $c \in\,]x, y[$ such that

$$f(y) - f(x) = (y - x) f'(x).$$

or $$f(y) - f(x) = (y - x) \cdot \frac{1}{1+c^2} \qquad \left[\because f'(t) = \frac{1}{1+t^2}\right]$$

$\therefore$ $f(y) - f(x) \le (y - x)$. $[\because\ 1 + c^2 \ge 1]$

Similarly, $f(x) - f(y) \le (x - y)$, when y < x.

$\therefore$ $|f(x) - f(y)| = |x - y|$.

Hence, $|\tan^{-1} x - \tan^{-1} y| \le |x - y|\ \forall\ x, y \in R$.

Ex. Let f be differentiable on an interval I and suppose that f's is bounded on I. Prove that f is uniformly continuous on I.

Sol. Let x_1, x_2 be any two arbitrary points of I with $x_1 < x_2$. Suppose f'(x) is bounded on I, there exists some k > 0 such that

$$|f'(x)| \le k\ \forall\ x \in I. \quad ...(1)$$

Applying Lagrange's mean value theorem to f on $[x_1, x_2]$, we get

$$\frac{f(x_2)-f(x_1)}{x_2-x_1} = f'(c) \text{ for some } c \in]x_1, x_2[$$

or $\quad | f(x_2) - f(x_1) | = | x_2 - x_1 | | f'(c) | \le k | x_2 - x_1 |$, by (1).

Let $\varepsilon > 0$ be given and let $\delta = \varepsilon/k > 0$. Then

$$| f(x_2) - f(x_1) | < \varepsilon, \text{ when } | x_2 - x_1 | < \delta, \forall x_1, x_2 \in I.$$

Hence f is uniformly continuous on I.

Second Mean Value Theorem :

Cauchy's Mean Value Theorem : Statement:

If two functions f and g with domain [a, b] are such that both are

[C_1] continuous in [a, b]

[C_2] differentiable in (a, b), and

[C_3] $g'(x) \neq 0 \quad \forall x \in (a, b)$

then $\exists c \in (a, b)$ such that:

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

Proof. First we notice that $g(b) \neq g(a)$

Let, if $g(b) = g(a)$, then g would satisfy all the conditions of Rolle's theorem, particularly [R_3], then by the theorem, $\exists c \in (a, b)$, where $g'(c) = 0$ which contradicts the condition [C_3].

Therefore our assumption $g(a) = g(b)$ is false.

Hence $g(a) \neq g(b)$ ...(1)

Now define a new function ϕ, with domain [a, b] involving the given functions f and g as follows:

$$\phi(x) = f(x) + A\, g(x), \qquad ...(2)$$

where A is a constant to be determined such that $\phi(a) = \phi(b)$

or, $\quad f(a) + A\, f(b) = g(a) + A\, g(b) \qquad$ [by (2)]

$$\Rightarrow \quad A = -\frac{f(b)-f(a)}{g(b)-g(a)} \qquad ...(3)$$

which exists because $g(b) \neq g(a)$.

Hence the function ϕ is expressed as sum of two continuous and derivable functions, therefore ϕ is :

(i) continuous on [a, b]

(ii) differentiable on (a, b) and

(iii) $\phi(a) = \phi(b)$ [by A of (3)]

Thus the given function ϕ satisfies all the three conditions of Rolle's theorem. Therefore Rolle's theorem is applicable and accordingly there must be atleast one point $x = c$ in (a, b), where $\phi'(c) = 0$.

$\therefore \quad \phi'(c) = f'(c) + A\, g'(c) = 0$

$$\Rightarrow \quad A = -\frac{f'(c)}{g'(c)} \qquad ....(4)$$

By (3) and (4), $\quad \dfrac{f(b)-f(a)}{g(b)-g(a)} = \dfrac{f'(c)}{g'(c)} \qquad$ **Hence Proved.**

Important Special Case

When $g(x) = x$, it reduces to Lagrange's MV Theorem.

Geometrical interpretation of Cauchy's Mean value Theorem

Statement : There is an ordinate x = c between x = a and x = b such that the tangents at the points where x = c cut the graph of the function f(x) and $\frac{f(b)-f(a)}{g(b)-g(a)}$ g(x) are mutually parallel.

Another useful form of Cauchy's Mean Value Theorem:

Statement: If two functions f(x) and g(x) with domain [a, a + h] are such that both are:

(1) Continuous in [a, a + h],

(2) Differentiable in (a, a + h), and

(3) $g'(x) \neq 0, \quad x \in (a, a+h)$

then $\exists\ \theta \in (0, 1)$ such that :

$$\frac{f'(a+\theta h)}{g'(a+\theta h)} = \frac{f(a+h)-f(a)}{g(a+h)-g(a)},\ 0 < \theta < 1$$

If we put b = a + h, then any point c = a + θh, 0 < θ < 1 may be taken in the interval [a, a + h].

Generalised Mean Value Theorem

If three functions f, g and h defined in [a, b] are such that

(i) f, g and h are continuous in [a, b]

(ii) f, g and h are derivable in]a, b[.

Then there exists a real number c ∈]a, b[such that

$$\begin{vmatrix} f'(c) & g'(c) & h'(c) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{vmatrix} = 0.$$

Proof. Define a function ψ on [a, b] as follows :

$$\psi(x) = \begin{vmatrix} f(x) & g(x) & h(x) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{vmatrix}$$

or $\qquad y(x) = A\, f(x) + B\, g(x) + C\, h(x),$ ...(iii)

where A = g(a) h(b) – h(a) g(b) etc., B and C are constants.

Applying given conditions (i) and (ii) in (iii), it follows that ψ is continuous in [a, b] and derivable in]a, b[.

Also $\qquad \psi(a) = 0 = \psi(b).$

($\because$ two rows in ψ(x) become identical by putting x = a and x = b)

So $\qquad \psi(a) = \phi(b).$

Thus ψ satisfies all the conditions of Rolle's theorem and, therefore, there exists some c ∈]a, b[such that ψ'(c) = 0.

Now $$\psi'(x) = \frac{d}{dx}\begin{vmatrix} f(x) & g(x) & h(x) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{vmatrix} = \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{vmatrix}$$

Hence $$\psi'(c) = 0 \text{ gives } \begin{vmatrix} f'(c) & g'(c) & h'(c) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{vmatrix} = 0,\ c \in\ ]a, b[.$$

Remarks 1. If we take g(x) = x and h(x) = 1 in the Generalised Mean Value Theorem, we obtain Lagrange's mean value theorem.

2. If we take h(x) = 1 in the Generalised Mean Value Theorem we obtain Cauchy's mean value theorem.

Example : Verify the Cauchy's mean value theorem for :

(i) $f(x) = x^2$, $g(x) = x^3$ in [1, 2].

(ii) f(x) = sin x, g(x) = cos x in [$-\pi/2$, 0]

(iii) $f(x) = e^x$, $g(x) = e^{-x}$ in [0, 1].

Solution. (i) Clearly $f(x) = x^2$, $g(x) = x^3$ are continuous in [1, 2] and derivable in]1, 2[. Further $g'(x) = 3x^2 \neq 0 \ \forall \ x \in]1, 2[$. Thus the conditions of the Cuachy's mean value theorem are satisfied and so there exists some point c ∈]1, 2[such that

$$\frac{f(2)-f(1)}{g(2)-f(1)} = \frac{f'(c)}{g'(c)}, 1 < c < 2$$

or $$\frac{4-1}{8-1} = \frac{2c}{3c^2} \text{ or } \frac{3}{7} = \frac{2}{3c}$$

or $$c = \frac{14}{9} = 1.55 \in]1, 2[.$$

Hence Cauchy's mean value theorem is verified.

(ii) Clearly f(x) = sin x, g(x) = cos x are continuous in [$-\pi/2$, 0] and derivable in]$-\pi/2$, 0[. Further g'(x) = − sin x ≠ 0 for all x ∈]$-\pi/2$, 0 [. Thus the conditions of the Cauchy's mean value theorem are satisfied and so there exists some point c ∈]$-\pi/2$, 0 [such that

$$\frac{f(0)-f(-\pi/2)}{g(0)-g(-\pi/2)} = \frac{f'(c)}{g'(c)} \text{ or } \frac{0+1}{1-0} = \frac{\cos c}{-\sin c}$$

or tan c = −1, which gives c = $-\pi/4$ ∈]$-\pi/2$, 0[.

Hence Cauchy's mean value theorem is verified.

(iii) Clearly $f(x) = e^x$ and $g(x) = e^{-x}$ satisfy the conditions of the Cauchy's mean value theorem and so there exists some c ∈]0, 1[such that

$$\frac{f(1)-f(0)}{g(1)-g(0)} = \frac{f'(c)}{g'(c)} \text{ or } = \frac{e-1}{e^{-1}-1} = \frac{e^c}{-e^{-c}}$$

or $$\frac{e(e-1)}{1-e} = -e^{2c} \text{ or } e^{2c} = e^1$$

or $$2c = 1 \quad \text{or} \quad c = \frac{1}{2} \in]0, 1[.$$

Hence Cauchy's mean value theorem is verified.

Example : Show that

$$\frac{\sin\alpha - \sin\beta}{\cos\beta - \cos\alpha} = \cot\theta, \text{ where } 0 < \alpha < \theta < \beta < \frac{\pi}{2}.$$

Solution. Let f(x) = sin x, g(x) = cos x ∀ x ∈ [α, β] and derivable in]α, β[.

Further g'(x) = − sin x ≠ 0 ∀ x ∈]a, b[∈]0, $\pi/2$[.

By Cauchy's mean value theorem,

$$\frac{f(\beta)-f(\alpha)}{g(\beta)-g(\alpha)}=\frac{f'(\theta)}{g'(\theta)}, \text{ for some } \theta \in]\alpha, \beta[.$$

i.e.,
$$\frac{\sin\beta-\sin\alpha}{\cos\beta-\cos\alpha}=\frac{\cos\theta}{-\sin\theta}, \quad \alpha<\theta<\beta.$$

Hence
$$\frac{\sin\alpha-\sin\beta}{\cos\beta-\cos\alpha} = \cot\theta, \quad \text{where } 0 < \alpha < \theta < \beta < \frac{\pi}{2}.$$

Example : Let the function f be continuous in [a, b] and derivable in]a, b[. Show that there exists a number c in]a, b[such that

$$2c\,[f(a) - f(b)] = f'(c)\,[a^2 - b^2].$$

Solution. Let $g(x) = x^2 \quad \forall \; x \in [a, b],\ a < b$.

Then f and g are continuous in [a, b] and derivable in]a, b[. Further $g'(x) = 2x \notin]a, b[$. By Cauchy's mean value theorem,

$$\frac{f(b)-f(a)}{g(b)-g(a)}=\frac{f'(c)}{g'(c)} \text{ for some } c \in]a, b[$$

i.e.,
$$\frac{f(b)-f(a)}{b^2-a^2}=\frac{f'(c)}{2c}.$$

Hence $2c\,[f(a) - f(b)] = f'(c)[a^2 - b^2],\ c \in]a, b[$

Example : If f' and g' exist for all $x \in [a, b]$ and if $g'(x) \neq 0\ \forall\ x \in]a, b[$, then prove that for some $c \in]a, b[$

$$\frac{f(c)-f(a)}{g(b)-g(c)}=\frac{f'(c)}{g'(c)}$$

Solution. Define a function ψ on [a, b] as follows :

$$\psi(x) = f(x)\,g(x) - f(a)\,g(x) - g(b)\,f(x),\ \forall\ x \in]a, b[.$$

Since f and g are derivable in [a, b], so ψ is continuous in [a, b] and derivable in]a, b[.

Also $\quad \psi(a) = \psi(b) = -f(a)\,g(b)$.

Since ψ satisfied all the conditions of Rolle's Theorem, therefore, there exists some $c \in]a, b[$ such that $\psi'(c) = 0$. We have

$$\psi'(x) = f'(x)\,g(x) + f(x)\,g'(x) - f(a)\,g'(x) - g(b)\,f'(x).$$

$\therefore \quad \psi'(c) = 0$ gives us

$$f'(c)\,g(c) + f(c)\,g'(c) - f(a)\,g'(c) - g(b)\,f'(c) = 0$$

or $\quad g'(c)\,\{f(c) - f(a)\} = f'(c)\,\{g(b) - g(c)\}$.

Hence
$$\frac{f(c)-f(a)}{g(b)-g(c)}=\frac{f'(c)}{g'(c)}$$

Example : If a function f is such that its derivative f' is continuous on [a, b] and derivable on]a, b[, then show that there exists a number c between a and b such that

$$f(b) = f(a) + (b - a)\,f'(a) + \frac{1}{2}(b - a)^2\,f''(c).$$

Solution. Define a function ϕ on [a, b] as follows :

$$\phi(x) = f(b) - f(x) - (b - x)\, f'(x) - (b - x)^2 A, \qquad ...(1)$$

where A is a constant to be determined by

$$\phi(a) = \phi(b) \qquad ...(2)$$

i.e., $f(b) - f(a) - (b - a)\, f'(a) - (b - a)^2 A = 0$

i.e., $f(b) = f(a) + (b - a)\, f'(a) + (b - a)^2 A.$ ...(3)

Since f' is continuous on [a, b], so f is also continuous on [a, b].

Also $(b - x)$, $(b - x)^2$ are continuous on [a, b].

Thus by (1), ϕ is continuous.

Thus ϕ satisfies the conditions of Rolle's Theorem and so there exists some point $c \in]a, b[$ such that $\phi'(c) = 0$. ...(4)

From (1), $\phi'(x) = -f'(x) - \{-f'(x) + (b - x)\, f''(x)\} + 2(b - x)A$

or $\phi'(x) = -(b - x)\, f''(x) + 2(b - x)\, A.$ ...(5)

$\therefore$ $f'(c) = 0 \quad \Rightarrow \quad -(b - c)\, f''(c) + 2(b - c)\, A = 0$

$$\Rightarrow \quad A = \frac{1}{2} f''(c), \text{ since } a < c < b \Rightarrow b - c \neq 0.$$

Substituting this value of A in (3), the result is proved.

Example : If f' is continuous on [a, a + h] and derivable on] a, a + h [, then prove that there exists a real number c between a and a + h such that

$$f(a + h) = f(a) + hf'(a) + \frac{h^2}{2} f''(c).$$

Solution. Define a function ϕ on [a, a + h] as follows :

$$\phi(x) = f(x) + (a + h - x)\, f'(x) + \frac{1}{2}(a + h - x)^2 A. \qquad ...(1)$$

where A is a constant to be determined by $\phi(a) = \phi(a + h)$

i.e., $f(a) + hf'(a) + \frac{h^2}{2} A = f(a + h).$ ...(2)

Since f' is continuous on [a, a + h], f and f' are continuous on [a, a + h]. Further $(a + h - x)$, $(a + h - x)^2$ are also continuous on [a, a + h] and so by (1), ϕ is continuous on [a, a + h]. Since f' is derivable on]a, a + h[, so by (1), ϕ is derivable on]a, a + h[. Thus ϕ satisfies all the conditions of Rolle's Theorem and so there exists some $c \in]\, a, a + h\, [$ such that

$\therefore$ $f'(c) = 0$ ...(3)

From (1), $\phi'(x) = f'(x) - f'(x) + (a + h - x)\, f''(x) - (a + h - x)\, A.$

$\therefore$ $0 = \phi'(c) = (a + h - c)\, [f''(c) - A]$ ...(4)

$\Rightarrow$ $f''(c) - A = 0$, since $c \in]a, a + h[$ means $a + h - c \neq 0$.

Putting $A = f''(c)$ in (2), we obtain

$$f(a + h) = f(a) + h\, f'(a) + \frac{1}{2} h^2 f''(c).$$

Example : Let f and g be two functions defined and continuous on [a, b] and derivable on (a, b). Show that there exists some $c \in (a, b)$ such that

$$\begin{vmatrix} f(a) & f(b) \\ g(a) & g(b) \end{vmatrix} = (b - a) \begin{vmatrix} f(a) & f'(c) \\ g(a) & g'(c) \end{vmatrix}.$$

Solution. We define a function ϕ on [a, b] as follows :

$$\phi(x) = \begin{vmatrix} f(a) & f(x) \\ g(a) & g(x) \end{vmatrix} - \left(\frac{x-a}{b-a}\right)\begin{vmatrix} f(a) & f(b) \\ g(a) & g(b) \end{vmatrix}. \quad ...(1)$$

By the given hypothesis, ϕ is continuous on [a, b] and derivable on (a, b). Also $\phi(a) = 0 = \phi(b)$. Thus ϕ satisfies the conditions of Rolle's theorem and so there exists some point $c \in (a, b)$ such that $\phi'(c) = 0$.

From (1), we have

$$\phi'(x) = \begin{vmatrix} f(a) & f'(x) \\ g(a) & g'(x) \end{vmatrix} - \frac{1}{(b-a)}\begin{vmatrix} f(a) & f(b) \\ g(a) & g(b) \end{vmatrix}.$$

$$\therefore \quad 0 = \phi'(c) = \begin{vmatrix} f(a) & f'(c) \\ g(a) & g'(c) \end{vmatrix} - \frac{1}{(b-a)}\begin{vmatrix} f(a) & f(b) \\ g(a) & g(b) \end{vmatrix}.$$

Hence
$$\begin{vmatrix} f(a) & f(b) \\ g(a) & g(b) \end{vmatrix} = (b-a)\begin{vmatrix} f(a) & f'(c) \\ g(a) & g'(c) \end{vmatrix}.$$

Example : If f" be continuous on [a, b] and derivable on]a, b[, then prove that

$$f(b) - f(a) - \frac{1}{2}(b-a)\,\{f'(a) + f'(b)\} = -\frac{(b-a)^3}{12} f'''(d),$$

for some real number d between a and b

Solution. Define a function g on [a, b] as follows ;

$$g(x) = f(x) - f(a) - \frac{1}{2}(x-a)\,\{f'(a) + f'(x)\} + A(x-a)^3,$$

where A is a constant to be suitably chosen.

Since f" is continuous on [a, b], so f and f' are continuous on [a, b].

Thus g is continuous on [a, b].

Similarly, g is derivable on]a, b[.

Let A be chosen such that g(a) = g(b).

$$\therefore \quad 0 = f(b) - f(a) - \frac{1}{2}(b-a)\,\{f'(a) + f'(b)\} + A(b-a)^3. \quad ...(1)$$

Since the function g satisfies all the conditions of Rolle's theorem [a, b], therefore, there exists a real number c between a and b such that

$$g'(c) = 0$$

We have
$$g'(x) = f'(x) - \frac{1}{2}\{f'(a) + f'(x)\} - \frac{1}{2}(x-a)\,f''(x) + 3A\,(x-a)^2$$

or
$$g'(x) = \frac{1}{2}\{f'(x) - f'(a)\} - \frac{1}{2}(x-a)\,f''(x) + 3A\,(x-a)^2.$$

Thus g'(c) = 0 implies that

$$\frac{1}{2}\{f'(c) - f'(a)\} - \frac{1}{2}(c-a)\,f''(c) + 3A(c-a)^2 = 0. \quad ...(2)$$

Let h be a function defined on [a, b] as follows :

$$h(x) = \frac{1}{2}\{f'(x) - f'(a)\} - \frac{1}{2}(x-a)\,f''(x) + 3A\,(x-a)^2 \quad ...(3)$$

Then (i) h is continuous on [a, c],

(ii) h is derivable on]a, c[,

(iii) $h(c) = 0$ and $h(a) = 0$, by (2) and (3); so that $h(a) = h(c)$.

Since the function h satisfies all the conditions of Rolle's theorem it [a, c], therefore, there exists a real number $d(a < d < c < b)$, such that

$$h'(d) = 0.$$

From (3), $$h'(x) = \frac{1}{2}f''(x) - \frac{1}{2}f''(x) - \frac{1}{2}(x-a)\, f'''(x) + 6\,A\,(x-a)$$

$$\therefore \quad 0 = h'(d) = -\frac{1}{2}(d-a)\, f'''(d) + 6A\,(d-a).$$

i.e., $$A = f'''(d)/12, \text{ since } (d-a) \neq 0. \qquad \ldots(4)$$

From (1) and (4), we have

$$f(b) - f(a) - \frac{1}{2}(b-a)\,\{f'(a) + f'(b)\} = -\frac{1}{12}(b-a)^3\, f'''(d).$$

Example : Assuming that $f''(x)$ exists for all x in [a, b], show that

$$f(c) - f(a)\frac{b-c}{b-a} - f(b)\frac{c-a}{b-a} - \frac{1}{2}(c-a)(c-b)\, f''(\xi) = 0 \qquad \ldots(1)$$

where c and ξ both lie in [a, b].

Solution. The equation (1) can be rewritten as

$$f(c)(b-a) - f(a)(b-c) - f(b)(c-a) - \frac{1}{2}(b-a)(c-a)(c-b)\, f''(\xi) = 0$$

or $$f(a)(c-b) - f(b)(c-a) - f(c)(b-a) - \frac{1}{2}(a-b)(b-c)(c-a)\, f''(\xi) = 0$$

or $$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ f(a) & f(b) & f(c) \end{vmatrix} - \frac{1}{2}f''(\xi)\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = 0. \qquad \ldots(2)$$

We shall prove (2) instead of (1).

The relation (2) helps us to define a function F(x) on [a, b] as follows:

$$F(x) = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ f(a) & f(b) & f(x) \end{vmatrix} - A\begin{vmatrix} 1 & 1 & 1 \\ a & b & x \\ a^2 & b^2 & x^2 \end{vmatrix} \qquad \ldots(3)$$

where A is a constant such that $F(c) = 0$. ...(4)

From (3), $F(a) = 0$, $F(b) = 0$.

Thus $F(a) = F(c)$ and $F(c) = F(b)$, $a < c < b$. ...(5)

Since f'' exists in [a, b], therefore, f and f' are derivable and continuous in [a, b]l an hence derivable on continuous in each of the intervals [a, c] and [c, b]. From (3) and (5), it follows that F satisfies the conditions of Rolle's theorem in each of the intervals [a, c] and [c, b]. Consequently,

$$F'(\xi_1) = 0, \text{ for some } \xi_1 \in\,]a, c[,$$

and $$F'(\xi_2) = 0, \text{ for some } \xi_2 \in\,]c, b[.$$

From (3),

$$F'(x) = \begin{vmatrix} 1 & 1 & 0 \\ a & b & 1 \\ f(a) & f(b) & f'(x) \end{vmatrix} - A \begin{vmatrix} 1 & 1 & 0 \\ a & b & 1 \\ a^2 & b^2 & 2x \end{vmatrix} \quad ...(6)$$

As stated earlier, f' is derivable and continuous in [a, b]. So by (6), F' is derivable and continuous in $[\xi_1, \xi_2] \subset [a, b]$ and $F'(\xi_1) = F'(\xi_2)$. Thus F' satisfies the conditions of Rolle's theorem in $[\xi_1, \xi_2]$ and so there exists $x \in]x_1, x_2[\subset [a, b]$ such that

$F''(\xi) = 0$

From (6), $$F''(x) = \begin{vmatrix} 1 & 1 & 0 \\ a & b & 0 \\ f(a) & f(b) & f''(x) \end{vmatrix} - A \begin{vmatrix} 1 & 1 & 0 \\ a & b & 0 \\ a^2 & b^2 & 2 \end{vmatrix}$$

$$= f''(x)(b-a) - 2A(b-a)$$

$\therefore \quad F''(\xi) = 0 \Rightarrow f''(\xi) - 2A = 0 \Rightarrow A = -\frac{1}{2} f''(\xi). \qquad (\because\ a \neq b)$

Substituting $A = \frac{1}{2}f''(\xi)$ in F(c) = 0 and using (3), we obtain

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ f(a) & f(b) & f(c) \end{vmatrix} - \frac{1}{2}f''(\xi) \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = 0.$$

which proves (2).

Example : If f(0) = 0 and f"(x) exists on $[0, \infty[$, show that

$$f'(x) - \frac{f(x)}{x} = \frac{1}{2} x f''(\xi),\ 0 < \xi < x, \quad ...(1)$$

and deduce that if f"(x) is positive for positive values of x, then f(x)/x strictly increases in $]0, \infty[$.

Solution. The relation (1) can be written as

$$f(x) - x f'(x) + \frac{1}{2}x^2 f''(\xi) = 0.$$

Define a function F as follows :

$$F(x) = f(x) - x f'(x) + \frac{1}{2}Ax^2, \quad ...(2)$$

where A is a constant such that F(c) = 0, where c > 0.

We have $\quad F(0) = f(0) = 0$ and $F(c) = 0. \quad \therefore \quad F(0) = F(c)$.

Thus F satisfies the conditions of Rolle's theorem on [0, c].

Therefore, there exists ξ such that $0 < \xi < c$ and $F'(\xi) = 0$.

From (2) $\quad F'(x) = -x f''(x) + Ax.$

$\therefore \quad F'(x) = 0 \Rightarrow A = f''(\xi).$

From (2), $\quad F(c) = 0 \Rightarrow f(c) - cf'(c) + \frac{1}{2}Ac^2 = 0$

$\Rightarrow$ $\quad f'(c) - \dfrac{f(c)}{c} = \dfrac{1}{2} c\, f''(\xi)\ \forall\ c > 0.$

Hence $\quad f'(x) - \dfrac{f(x)}{x} = \dfrac{1}{2} x\, f''(\xi),\ 0 < \xi < x.$

(ii) Let $\quad G(x) = f(x)/x,\quad$ whenever $x > 0.$

$\therefore \quad G'(x) = \dfrac{x f'(x) - f(x)}{x^2} = \dfrac{1}{2} f''(\xi) > 0$, as $f''(x) > 0\ \forall\ x > 0$

Thus $\quad G'(x) > 0\ \forall\ x > 0.$ ...(3)

If x_1 and x_2 be any two positive real numbers such that $x_2 > x_1$, then by applying Lagrange's mean value theorem to G in $[x_1, x_2]$, we get

$$G(x_2) - G(x_1) = (x_2 - x_1)\, G'(\theta),\ \theta \in]x_1, x_2[$$

Since $G'(\theta) > 0$, by (3), and $x_2 - x_1 > 0$, so

$$G(x_2) - G(x_1) > 0 \text{ i.e., } G(x_2) > G(x_1) \text{ for } x_2 > x_1 > 0.$$

Hence $G(x) = f(x)/x$ is strictly increasing in $]0, \infty[$.

Example : A twice differentiable function f is such that

$$f(a) = f(b) = 0 \text{ and } f(c) > 0 \text{ for } a < c < b.$$

Prove that there is at least one value ξ between a and b for which

$$f''(\xi) < 0.$$

Solution. Since f'' exists in [a, b], therefore f and f' both exist and are continuous in [a, b] and as well as in [a, c] and [c, b], since $a < c < b$. Applying Lagrange's mean value theorem to f on [a, c] and [a, b], we get

$$\frac{f(c) - f(a)}{c - a} = f'(\xi_1),\ a < \xi_1 < c,$$

and $$\frac{f(b) - f(c)}{b - c} = f'(\xi_2),\ c < \xi_2 < b, \text{ respectively.}$$

Since $f(a) = f(b) = 0$, the above relationship become

$$f'(\xi_1) = \frac{f(c)}{c - a},\ f'(\xi_2) = -\frac{f(c)}{b - c},\ a < \xi_1 < c < \xi_2 < b. \quad ...(1)$$

Since f' is continuous and derivable on $[\xi_1, \xi_2]$, so on applying Lagrange's mean value theorem of f' on $[\xi_1, \xi_2]$, we obtain

$$\frac{f'(\xi_2) - f'(\xi_1)}{\xi_2 - \xi_1} = f''(x), \quad \text{for } \xi_1 < \xi < \xi_2. \quad ...(2)$$

From (1) and (2), we get

$$f''(\xi) = -\frac{f(c)}{\xi_2 - \xi_1}\left(\frac{1}{b - c} + \frac{1}{c - a}\right)$$

$$= \frac{-(b - a) f(c)}{(\xi_2 - \xi_1)(b - c)(c - a)} < 0,$$

since $f(c) > 0$, $a < c < b$ and $\xi_1 < \xi_2$.

Hence $f''(\xi) < 0$ for some ξ, $a < \xi < b$.

Example : If f" be defined on [a, b] and if $| f''(x) | \le M$ for all in [a, b], then prove that

$$\left| f(b) - f(a) - \frac{1}{2}(b - a)\{f'(a) + f'(b)\} \right| \le \frac{1}{2}(b - a)^2 M.$$

Solution. Define a function ϕ on [a, b] as follows :

$$\phi(x) = f(x) - f(a) - \frac{1}{2}(x - a)\{f'(a) + f'(x)\} + A(x - a)^2, \qquad ...(1)$$

where A is a constant to be chosen such that $\phi(b) = 0$ ...(2)

i.e., $f(b) - f(a) - \frac{1}{2}(b - a)\{f'(a) + f'(b)\} + A(b - a)^2 = 0.$ ...(3)

Since f" is defined on [a, b], so f and f' are both derivable on [a, b] i.e., ϕ is derivable on [a, b] and so ϕ is continuous on [a, b].

From (1) and (2), $\phi(a) = 0$, $\phi(b) = 0$ and so $\phi(a) = \phi(b)$.

By Rolle's Theorem, there exists some $c \in]a, b[$ such that

$$\phi'(c) = 0$$

From (1), $\phi'(x) = f'(x) - \frac{1}{2}\{f'(a) + f'(x)\} - \frac{1}{2}(x - a) f''(x) + 2A(x - a)$

$\therefore \quad \phi'(c) = 0 \Rightarrow f'(c) - f'(a) - (c - a) f''(c) + 4A(c - a) = 0.$ ...(4)

Applying Lagrange's Mean Value Theorem to f' on [a, c], we obtain

$$f'(c) - f'(a) = (c - a) f''(d) \text{ for some } d \in]a, c[.$$

Substituting in (4), we obtain

$$f''(d) - f''(c) + 4A = 0 \Rightarrow A = \frac{1}{4}[f''(c) - f''(d)].$$

Putting this value of A in (3), we obtain

$$f(b) - f(a) - \frac{1}{2}(b - a)\{f'(a) + f'(b)\} = \frac{1}{4}(b - a)^2[f''(d) - f''(c)].$$

Hence $\left| f(b) - f(a) - \frac{1}{2}(b - a)\{f'(a) + f'(b)\} \right|$

$$\le \frac{1}{4}(b - a)^2[\,|f''(d)| + |f''(c)|\,]$$

$$\le \frac{1}{4}(b - a)^2(M + M) \qquad \{\because |f''(x)| \le M \ \forall\ x \in [a, b]\}$$

$$= \frac{1}{4}(b - a)^2 M.$$

Example : Show that 'θ' which occurs in the Lagrange's mean value theorem tends to the limit $\frac{1}{2}$ as $h \to 0$, provided f" is continuous.

Solution. We may take f" to be continuous in [a, a + h].

So by Example we have

$$f(a + h) = f(a) + hf'(a) + \frac{h^2}{2} f''(a + \theta_1 h), \text{ where } 0 < \theta_1 < 1. \qquad ...(1)$$

Applying Lagrange' mean value theorem to f in [a, a + h], we get

$$f(a + h) - f(a) = hf'(a + \theta h),\ 0 < \theta < 1. \quad ...(2)$$

From (1) and (2), we obtain

$$f'(a + \theta h) = f'(a) + \frac{1}{2}hf''(a + \theta_1 h). \quad ...(3)$$

Applying Lagrange's mean value theorem to the function f' in [a, a + θh], we obtain

$$f'(a + \theta h) - f'(a) = \theta h\ .\ f''(a + \theta_2\theta h),\ 0 < \theta_2 < 1. \quad ...(4)$$

From (3) and (4), we get

$$\theta h.\ f''(a + \theta_2\theta h) = \frac{1}{2}h\ f''(a + \theta_1 h)$$

$$\Rightarrow \quad \theta = \frac{1}{2}.\frac{f''(a+\theta_1 h)}{f''(a+\theta_2\theta h)}.$$

$$\therefore \quad \lim_{h\to 0}\theta = \frac{1}{2}.\frac{f''(a)}{f''(a)} = \frac{1}{2},$$ since f'' continuous in [a, a + h].

Hence $\theta \to \frac{1}{2}$ as $h \to 0$.

Ex : If f''(x) exists ∀ x ∈ [a, b] and

$$\frac{f(c)-f(a)}{c-a} = \frac{f(b)-f(c)}{b-c},\ \text{where } c \in\ ]a, b[. \quad ...(1)$$

Then there exists some $\xi \in\ ]a, b[$ such that $f''(\xi) = 0$.

Sol. Since f''(x) exists ∀ x [a, b], therefore, f and f' exist in [a, b] and are continuous in [a, b].

Applying Lagrange's Mean Value Theorem to the function f on each of the interval [a, c] and [c, d], we get

$$\frac{f(c)-f(a)}{c-a} = f'(\xi_1) \text{ for some } \xi_1 \in\ ]a, b[,$$

and
$$\frac{f(b)-f(c)}{b-c} = f'(\xi_2), \text{ for some } \xi_2 \in\ ]c, b[.$$

Using (1), it follows that $f'(\xi_1) = f'(\xi_2)$. Further.

(i) f' is continuous in $[\xi_1, \xi_2]$, (ii) f' derivable in $]\xi_1, \xi_2[$ and (iii) $f'(\xi_1) = f'(\xi_2)$. So by Rolle's Theorem as applied to f' in $[\xi_1, \xi_2]$, we obtain

$$f''(\xi) = 0 \text{ for some } \xi \in\ ]\xi_1, \xi_2[\ \subset\ ]a, b[.$$

Ex : If f''(x) > 0 for all x ∈ R, then show that

$$f\left[\frac{1}{2}(x_1 + x_2)\right] \le \frac{1}{2}[f(x_1) + f(x_2)], \quad ...(1)$$

for every pair of real numbers x_1 and x_2.

Sol. Let $x_1, x_2 \in R$ be arbitrary. If $x_1 = x_2$, then the given relation (1) because equality and so the result follows. We may suppose that $x_1 < x_2$. Since f''(x) exists ∀ x ∈ R, if satisfies the conditions of Lagrange's mean value theorem in each of the intervals $\left[x_1, \frac{1}{2}(x_1 + x_2)\right]$ and $\left[\frac{1}{2}(x_1 + x_2), x_2\right]$. Consequently,

$$f\left[\frac{1}{2}(x_1+x_2)\right]-f(x_1)=\left[\frac{1}{2}(x_1+x_2)-x_1\right]f'(c_1), x_1<c_1<\frac{1}{2}(x_1+x_2)$$

and $\quad f(x_2) - f\left[\frac{1}{2}(x_1+x_2)\right]=\left[x_2-\frac{1}{2}(x_1+x_2)\right] f'(c_2),\ \frac{1}{2}(x_1 + x_2) < c_2 < x_2.$

On subtracting the corresponding sides of these equations, we get

$$2f\left[\frac{1}{2}(x_1+x_2)\right]-f(x_1)-f(x_2)=\frac{1}{2}(x_2-x_1)\left[f'(c_1)-f'(c_2)\right]. \quad ...(2)$$

Since $f''(x)$ exists $\forall\ x \in R$, f' satisfies the conditions of Lagrange's mean value theorem in the interval $[c_1, c_2]$ and so

$f'(c_2) - f'(c_1) = (c_2 - c_1) f''(t)$, for some $t \in]c_1, c_2[$.

As given $\quad f''(t) > 0$ and $c_2 > c_1$, so $(c_2 - c_1) f''(t) > 0$.

Thus $\quad f'(c_2) - f'(c_1) > 0$ i.e., $f'(c_1) - f'(c_2) < 0$.

Also $\quad x_2 > x_1 \Rightarrow x_2 - x_1 > 0$.

Using these inequalities in (2), we get

$$2f\left[\frac{1}{2}(x_1+x_2)\right]-f(x_1)-f(x_2)<0.$$

Hence $\quad f\left[\frac{1}{2}(x_1+x_2)\right]<\frac{1}{2}[f(x_1) + f(x_2)].$

Ex : If f', g' are continuous on $[a - h, a + h]$ and derivable on $]a - h, a + h[$, then prove that

$$\frac{f(a+h)-2f(a)+f(a-h)}{g(a+h)-2g(a)+g(a-h)}=\frac{f''(d)}{g''(d)}$$

for some $d \in]a - h, a + h[$, provided $g(a + h) - 2g(a) + g(a - h) \neq 0$ and $g''(t) \neq 0$ for each $t \in]a - h, a + h[$.

Sol. We define a function ϕ on $[a - h, a + h]$ as follows :

$$\phi(x) = f(x) + Ag(x) + Bx + C, \quad ...(1)$$

where A, B, C are constants to be determined by

$$\phi(a - h) = \phi(a) = \phi(a + h) = 0. \quad ...(2)$$

Since f and g are continuous on $[a - h, a + h]$ and $Bx + C$ is also continuous on $[a - h, a + h]$, so by (1), ϕ is continuous on $[a - h, a + h]$. Similarly, ϕ is derivable on $]a - h, a + h[$. By virtue of (2), ϕ satisfies all the conditions of Rolle's theorem in each of the intervals $[a - h, a]$ and $[a, a + h]$. Consequently.

$\phi'(t_1) = 0$, for some $t_1 \in]a - h, a[$,

and $\quad \phi'(t_2) = 0$, for some $t_2 \in]a, a + h[$.

For any $x \in]a - h, a + h[$, we have by (1),

$$\phi'(x) = f(x) + Ag'(x) + B. \quad ...(3)$$

Since f' and g' are continuous on $[a - h, a + h]$, so by (3), ϕ' is continuous on $[a - h, a + h]$. Similarly, ϕ' is derivable on $]a - h, a + h[$. Also $\phi'(t_1) = \phi'(t_2) = 0$. Applying Rolle's theorem to ϕ' on $[t_1, t_2]$, there exists some $d \in]t_1, t_2[$ such that $\phi''(d) = 0$ for some $d \in]t_1, t_2[$. Using (3),

$$f''(d) + Ag''(d) = 0 \Rightarrow A = - f''(d)/g''(d). \quad ...(4)$$

From (2), $\quad \phi(a + h) - 2\phi(a) + \phi(a - h) = 0$

or $\quad [f(a + h) - 2 f(a) + f(a - h)] + A[g(a + h) - 2g(a) + g(a - h)]$
$$+ [B(a + h) + C - 2 (Ba + C) + B(a - h) + C] = 0$$

or $$-A = \frac{f(a+h)-2f(a)+f(a-h)}{g(a+h)-2g(a)+g(a-h)}. \qquad ...(5)$$

From (4) and (5), we get the desired result.

Ex : Show that

(a) $\dfrac{x}{1+x} < \log(1 + x) < x,\ x > 0.$

(b) $x - \dfrac{x^2}{2} < \log(1 + x) < x - \dfrac{x^2}{2(1+x)},\ x > 0.$

(c) $\dfrac{x^2}{2(1+x)} < x - \log(1 + x) < \dfrac{x^2}{2},\ x > 0.$

Sol. Let $f(x) = \log(1 + x) - \dfrac{x}{1+x}$, so that $f(0) = 0$.

We shall show that $f(x) > 0$ for $x > 0$.

Now $\quad f'(x) = \dfrac{1}{1+x} - \dfrac{1}{(1+x)^2} = \dfrac{x}{(1+x)^2} > 0$, for $x > 0$.

Thus $f(x)$ is an increasing function $\forall\ x > 0$

$\Rightarrow \quad f(x) > f(0)$, for $x > 0$

$\Rightarrow \quad f(x) > 0$, for $x > 0 \qquad (\because\ f(0) = 0)$

$\Rightarrow \quad \log(1 + x) - \dfrac{x}{1+x} > 0,\ x > 0$

$\therefore \quad \dfrac{x}{1+x} < \log(1 + x),\ x > 0.$

Let $\quad \phi(x) = x - \log(1 + x)$, so that $\phi(0) = 0$.

$\therefore \quad \phi'(x) = 1 - \dfrac{1}{1+x} = \dfrac{x}{1+x} > 0$, for $x > 0$.

Thus $\quad \phi(x)$ is an increasing function of x, for $x > 0$

$\Rightarrow \quad \phi(x) > f(0)$, for $x > 0$

$\Rightarrow \quad \phi(x) > 0 \qquad (\because\ \phi(0) = 0)$

$\Rightarrow \quad x - \log(1 + x) > 0$, for $x > 0$

$\Rightarrow \quad \log(1 + x) < x$, for $x > 0$.

Hence $\dfrac{x}{1+x} < \log(1 + x) < x,\ x > 0$

(b) Let $f(x) = \log(1 + x) - \left(x - \dfrac{x^2}{2}\right)$, $f(0) = 0$.

$\therefore \quad f'(x) = \dfrac{1}{1+x} - 1 + x = \dfrac{x^2}{1+x} > 0. \qquad (\because\ x > 0)$

$\Rightarrow$ f(x) is an increasing function of x, for $x > 0$ ...(2)

$\Rightarrow$ $f(x) > f(0)$, for $x > 0 \Rightarrow f(x) > 0$ $(\because f(0) = 0)$

$\Rightarrow$ $\log(1 + x) - \left(x - \frac{x^2}{2}\right) > 0$, for $x > 0$

$\therefore$ $x - \frac{x^2}{2} < \log(1 + x)$, $x > 0$.

Let $\phi(x) = x - \frac{x^2}{2(1+x)} - \log(1 + x)$, so that $\phi(0) = 0$.

$\therefore$ $\phi'(x) = 1 - \frac{1}{2}\left\{\frac{2x + x^2}{(1+x)^2}\right\} - \frac{1}{1+x}$

$$= \frac{x^2}{2(1+x)^2} > 0 \qquad (\because x > 0)$$

$\Rightarrow$ $\phi(x)$ is an increasing function, for $x > 0$

$\Rightarrow$ $\phi(x) > \phi(0)$, for $x > 0$

$\Rightarrow$ $f(x) > 0$ $(\because f(0) = 0)$

$\Rightarrow$ $x - \frac{x^2}{2(1+x)} - \log(1 + x) > 0$, for $x > 0$

$\therefore$ $\log(1 + x) < x - \frac{x^2}{2(1+x)}$, for $x > 0$

Hence $x - \frac{x^2}{2} < \log(1 + x) < -\frac{x^2}{2(1+x)}$, for $x > 0$

(c) By part (b), we have

$$-x + \frac{x^2}{2} > -\log(1 + x) > \frac{x^2}{2(1+x)} - x, \ x > 0$$

or $\frac{x^2}{2} > x - \log(1 + x) > \frac{x^2}{2(1+x)}$, $x > 0$.

Hence $\frac{x^2}{2(1+x)} < x - \log(1 + x) < \frac{x^2}{2}$, $x > 0$.

Ex : Prove that $x < \sin^{-1} x < \frac{x}{\sqrt{1-x^2}}$, if $0 < x < 1$.

Sol. Let $f(x) = \sin^{-1} x - x$, so that $f(0) = 0$.

$\therefore$ $f'(x) = \frac{1}{\sqrt{1-x^2}} - 1 = \frac{1-\sqrt{1-x^2}}{\sqrt{1-x^2}}$

$\Rightarrow$ $f'(x) > 0$, for $0 < x < 1$

$\Rightarrow$ f is strictly increasing in $0 < x < 1$

$\Rightarrow$ $f(x) > f(0)$, for $0 < x < 1$ $(\because x > 0)$

$\Rightarrow$ f(x) > 0, for 0 < x < 1.

$\Rightarrow$ $\sin^{-1} x - x > 0$, for 0 < x < 1

$\Rightarrow$ $\sin^{-1} x > x$, for 0 < x < 1

$\Rightarrow$ $x < \sin^{-1} x$, for 0 < x < 1.

Let $g(x) = \dfrac{x}{\sqrt{1-x^2}} - \sin^{-1} x$, so that g(0) = 0.

$$\therefore \quad g'(x) = \frac{\sqrt{1-x^2} + \dfrac{x(-2x)}{2\sqrt{1-x^2}}}{(1-x^2)} - \frac{1}{\sqrt{1-x^2}}$$

$$= \frac{1}{(1-x^2)\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}} = \frac{x^2}{(1-x^2)^{3/2}}$$

$\Rightarrow$ g'(x) > 0, for 0 < x < 1

$\Rightarrow$ g is strictly increasing, for 0 < x < 1

$\Rightarrow$ g(x) > g(0), for 0 < x < 1 $(\because x > 0)$

$$\Rightarrow \quad \frac{x}{\sqrt{1-x^2}} > \sin^{-1} x, \text{ for } 0 < x < 1$$

$$\Rightarrow \quad \sin^{-1} x < \frac{x}{\sqrt{1-x^2}}, \text{ for } 0 < x < 1.$$

Ex : Using Lagrange's mean value theorem, show that

$$\frac{x}{1+x} < \log(1+x) < x,\ x > 0.$$

Sol. Let f(x) = log (1 + x) in [0, x], so that $f'(x) = \dfrac{1}{1+x}$.

Since f is continuous in [0, x] and derivable in]0, x[, so by Lagrange's mean value theorem, there exists some θ, $0 < \theta < 1$ such that

$$\frac{f(x)-f(0)}{x-0} = f'(\theta x) \qquad (\because\ ]0, x[\ =\ ]0x, 1x[,\ c = \theta x)$$

or $$\log(1+x) = \frac{x}{1+\theta x}. \qquad [\because f(0) = 0] \qquad ...(1)$$

Now $0 < \theta < 1$ and $x > 0 \Rightarrow \theta x < x$

$$\Rightarrow \quad 1 + \theta x < 1 + x \Rightarrow \frac{1}{1+\theta x} > \frac{1}{1+x}$$

$$\Rightarrow \quad \frac{x}{1+\theta x} > \frac{x}{1+x} \Rightarrow \frac{x}{1+x} < \frac{x}{1+\theta x}. \qquad ...(2)$$

Again $0 < \theta < 1$ and $x > 0 \Rightarrow y < 1 + \theta x$

$$\Rightarrow \quad \frac{1}{1+\theta x} < 1 \Rightarrow \frac{x}{1+\theta x} < x. \qquad ...(3)$$

From (2) and (3), we obtain

$$\frac{x}{1+x} < \frac{x}{1+\theta x} < x. \qquad ...(4)$$

From (1) and (4), we obtain

$$\frac{x}{1+x} < \log(1+x) < x.$$

Ex : Use the mean value theorem to prove

$$\frac{x}{1+x^2} < \tan^{-1} x < x, \text{ if } x > 0.$$

Sol. Let $f(x) = \tan^{-1} x$ in $[0, x]$

Then f satisfies the conditions of Lagrange's mean value theorem in $[0, x]$. Consequently, there exists some θ satisfying $0 < \theta < 1$ such that

$$\frac{f(x)-f(0)}{x-0} = f'(\theta x) \quad \left[f'(x) = \frac{1}{1+x^2} \text{ and } f(0) = 0\right]$$

or $$\tan^{-1} x = \frac{x}{1+\theta^2 x^2}. \qquad ...(1)$$

Now $0 < \theta < 1$ and $x > 0 \Rightarrow \theta^2 x^2 < x^2 \Rightarrow 1 + \theta^2 x^2 < 1 + x^2$

$$\Rightarrow \frac{x}{1+\theta^2 x^2} > \frac{x}{1+x^2} \Rightarrow \frac{x}{1+x^2} < \frac{x}{1+\theta^2 x^2} \qquad ...(2)$$

Again $0 < \theta < 1$ and $x > 0 \Rightarrow 1 < 1 + \theta^2 x^2$

$$\Rightarrow \frac{1}{1+\theta^2 x^2} < 1 \Rightarrow \frac{x}{1+\theta^2 x^2} < x. \qquad ...(3)$$

From (2) and (3), we obtain

$$\frac{x}{1+x^2} < \frac{x}{1+\theta^2 x^2} < x, \ x > 0. \qquad ...(4)$$

From (1) and (4), we obtain

$$\frac{x}{1+x^2} < \tan^{-1} x < x, \ x > 0.$$

Ex : Show that

$$\frac{1}{2}x^2 < x - \log(1+x) < \frac{1}{2}\frac{x^2}{(1+x)}, \text{ if } -1 < x < 0.$$

Sol. Let $f(x) = x - \log(1 + x) - \frac{1}{2}x^2$, so that $f(0) = 0$.

$$\therefore \quad f'(x) = 1 - \frac{1}{1+x} - x = -\left(\frac{x^2}{1+x}\right). \qquad ...(1)$$

Now $-1 < x < 0 \Rightarrow x + 1 > 0$. Thus, by (1),

$f'(x) < 0$, for $-1 < x < 0$

$\Rightarrow$ f is decreasing for $-1 < x < 0$

$\Rightarrow$ $f(x) > f(0)$, for $-1 < x < 0$

$\Rightarrow$ $f(x) > 0$, for $-1 < x < 0$

$\Rightarrow$ $\frac{1}{2}x^2 < x - \log(1 + x)$, for $-1 < x < 0$.

Let $g(x) = \frac{1}{2}\frac{x^2}{(1+x)}$ $x + \log(1 + x)$, so that $g(0) = 0$.

$\therefore$ $g'(x) = \frac{2x+x^2}{2(1+x)^2} - 1 + \frac{1}{1+x} = -\left\{\frac{x^2}{2(1+x)^2}\right\}$

$\Rightarrow$ $g'(x) < 0$, for $-1 < x < 0$

$\Rightarrow$ g is decreasing, for $-1 < x < 0$

$\Rightarrow$ $g(x) > g(0)$, for $-1 < x < 0$

$\Rightarrow$ $g(x) > 0$, for $-1 < x < 0$

$\Rightarrow$ $x - \log(1 + x) < \frac{1}{2}\frac{x^2}{(1+x)}$, $-1 < x < 0$

Ex : Prove that

$$x - \frac{x^2}{2} + \frac{x^3}{3(1+x)} < \log(1+x) < x - \frac{x^2}{2} + \frac{x^3}{3}, \text{ if } x > 0.$$

Sol. Let $f(x) = \log(1 + x) - \left[x - \frac{x^2}{2} + \frac{x^3}{3(1+x)}\right]$.

$\therefore$ $f'(x) = \frac{1}{1+x} - 1 + x - \frac{3x^2+2x^3}{3(1+x)^2} = \frac{x^3}{3(1+x)^2}$

$\Rightarrow$ $f'(x) > 0$, for $x > 0$

$\Rightarrow$ f is increasing, for $x > 0$

$\Rightarrow$ $f(x) > f(0)$, for $x > 0$

$\Rightarrow$ $f(x) > 0$ $(\because f(0) = 0)$

$\Rightarrow$ $x - \frac{x^2}{2} + \frac{x^3}{3(1+x)} < \log(1 + x)$, if $x > 0$.

Let $g(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \log(1 + x)$.

$\therefore$ $g'(x) = 1 - x + x^2 - \frac{1}{1+x} = \frac{x^3}{1+x}$

$\Rightarrow$ $g'(x) > 0$, for $x > 0$

$\Rightarrow$ g is increasing, for $x > 0$

$\Rightarrow$ $g(x) > g(0)$, for $x > 0$

$\Rightarrow$ $g(x) > 0$ $(\because g(0) = 0)$

$\Rightarrow$ $\log(1 + x) < x - \frac{x^2}{2} + \frac{x^3}{3}$, if $x > 0$.

Ex : If $0 < x < 1$, show that

$$2x < \log\frac{1+x}{1-x} < 2x\left(1 + \frac{1}{3}.\frac{x^2}{1-x^2}\right).$$

Sol. Let $f(x) = \log\frac{1+x}{1-x} - 2x$.

$$\therefore \quad f'(x) = \frac{1-x}{1+x}\left\{\frac{(1-x)+(1+x)}{(1-x)^2}\right\} - 2 = \frac{2}{1-x^2} - 2 = \frac{2x^2}{1-x^2}.$$

Clearly, $f'(x) > 0$, for $0 < x < 1 \Rightarrow f$ is increasing, for $0 < x < 1$.

In particular, $f(x) > f(0) = 0$ as $x > 0$

$$\therefore \quad \log\frac{1+x}{1-x} > 2x, \text{ for } 0 < x < 1.$$

Hence $\quad 2x < \log\frac{1+x}{1-x}$, for $0 < x < 1$.

Let $\quad g(x) = 2x\left(1+\frac{1}{3}\cdot\frac{x^2}{1-x^2}\right) - \log\frac{1+x}{1-x}$.

$$\therefore \quad g'(x) = 2 + \frac{2}{3}\left[\frac{(1-x^2).3x^2 + x^3.2x}{(1-x^2)^2}\right] - \frac{2}{1-x^2}$$

$$= \frac{6(1-x^2)^2 + 2(3x^2 - x^4) - 6(1-x^2)}{3(1-x^2)^2} = \frac{4x^4}{3(1-x^2)^2} > 0, \text{ if } 0 < x < 1$$

Thus $g(x)$ is increasing, for $0 < x < 1$. In particular,

$$g(x) > g(0) = 0 \Rightarrow g(x) > 0$$

Hence $\quad \log\frac{1+x}{1-x} < 2x\left(1+\frac{1}{3}\cdot\frac{x^2}{1-x^2}\right)$.

Ex : Prove that $\tan x > x$, whenever $0 < x < \pi/2$.

Sol. Let c be any real number such that $0 < c < \pi/2$.

Let $\quad f(x) = \tan x - x$ for all $x \in [0, c]$.

Then f is continuous as well as derivable on $[0, c]$.

Now $f'(x) = \sec^2 x - 1 = \tan^2 x > 0$, for $0 < x < c$.

Thus f is strictly increasing in $[0, c]$

$\Rightarrow \quad f(c) > f(0)$, for $c > 0$.

But $f(0) = 0$. Therefore $f(c) > 0 \Rightarrow \tan c - c > 0$.

Since c is any real number such that $0 < c < \pi/2$, therefore,

$$\tan x > x, \text{ whenever } 0 < x < \pi/2.$$

Ex : Show that $\frac{2}{\pi} < \frac{\sin x}{x} < 1$, $0 < x < \frac{\pi}{2}$.

Sol. Let f be defined as

$$f(x) = \begin{cases} \frac{\sin x}{x}, & \text{if } x \neq 0, \\ 1, & \text{if } x = 0. \end{cases} \quad \ldots(1)$$

Then f is continuous in $[0, \pi/2]$ and derivable in $]0, \pi/2[$.

We have $f'(x) = \dfrac{x\cos x - \sin x}{x^2}$. ...(2)

Let $g(x) = x \cos x - \sin x$ in $[0, \pi/2]$

$\therefore$ $g'(x) = \cos x - x \sin x - \cos x$

$= - x \sin x < 0$ in $]0, \pi/2[$.

$\Rightarrow$ g is strictly decreasing in $[0, \pi/2]$.

$\Rightarrow$ $g(x) < g(0) = 0$ in $[0, \pi/2]$ $(\because x > 0)$

Using in (2), it follows that $f'(x) < 0$ in $]0, \pi/2[$

$\Rightarrow$ f is strictly decreasing in $[0, \pi/2]$

$\Rightarrow$ $f(0) > f(x) > f(\pi/2)$, for $0 < x < \pi/2$

$\Rightarrow$ $1 > \dfrac{\sin x}{x} > \dfrac{1}{\pi/2}$, using (1)

Hence $\dfrac{2}{\pi} < \dfrac{\sin x}{x} < 1$, for $0 < x < \dfrac{\pi}{2}$.

Ex : Prove that

$$\frac{\tan x}{x} > \frac{x}{\sin x}, \text{ whenever } 0 < x < \frac{\pi}{2}.$$

Sol. $\dfrac{\tan x}{x} - \dfrac{x}{\sin x} = \dfrac{\tan x \sin x - x^2}{x \sin x}$

Since $x \sin x > 0$ for all x in $]0, \pi/2[$, therefore, we need to show that

$\tan x \sin x - x^2 > 0, \forall x \in]0, \pi/2[$.

Let c any real number in $]0, \pi/2[$.

Let $f(x) = \tan x \sin x - x^2 \ \forall x \in [0, c]$

Then f is continuous as well as derivable in $[0, c]$.

Now $f'(x) = \sec^2 x \sin x + \tan x \cos x - 2x$

$= \sin x (\sec^2 x + 1) - 2x$.

The form of $f'(x)$ is such that we cannot decide about its sign.

Let $g'(x) = \cos x (\sec^2 x + 1) + \sin x \,.\, 2 \sec^2 x \tan x - 2$

$= (\sqrt{\sec x} - \sqrt{\cos x})^2 + 2 \sin^2 x \sec^3 x$.

Since $g'(x) > 0$ for all $x \in]0, c[\Rightarrow$ g is strictly increasing in $[0, c]$

$\Rightarrow$ $g(x) > g(0)$, whenever $0 < x < c$. Since $g(0) = 0$, this means that $g(x) > 0$, whenever $0 < x < c \Rightarrow f'(x) > 0$, whenever $0 < x < c \Rightarrow$ f is strictly increasing in $[0, c] \Rightarrow f(c) > f(0) = 0 \Rightarrow f(c) > 0$.

$\Rightarrow$ $\tan c \sin c - c^2 > 0$

$\Rightarrow$ $\dfrac{\tan c}{c} - \dfrac{c}{\sin c} > 0$.

Since c is any point of $]0, \pi/2[$, it follows that

$$\frac{\tan x}{x} > \frac{x}{\sin x}, \text{ whenever } 0 < x < \frac{\pi}{2}.$$

Ex. Verify Lagranges MV Theorem for the function $f(x) = |x|$ in the interval $[-1, 2]$.

Sol. L_1. The given function is continuous in $[-1, 2]$, because

$$f(a) = |a|, f(a+0) = \lim_{h \to 0} |a+h| = |a|$$

and $\quad f(a-0)=\lim\limits_{h\to 0}|a-h|=|a|$

L_2. At the point x = 0 of the open interval (–1, 2)

$$f'(0+0)=\lim_{h\to 0}\frac{|0+h|-0}{h}=1$$

$$f'(0-0)=\lim_{h\to 0}\frac{|0-h|-0}{-h}=-1$$

Thus $\quad f'(0+0)\neq f'(0-0)$

Therefore the given function is not derivable in (–1, 2).

Hence Lagrange's theorem is not applicable for the given function.

3. TAYLOR'S THEOREM

Taylor's theorem with Lagrange's form of remainder;

Statement : If a function f with domain [a, a + h] is such that:

(i) f^{n-1} is continuous in [a, a + h]

(ii) f^{n} exists in (a, a + h)

then $\exists\ \theta \in (0, 1)$ such that :

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \ldots\ldots\ldots\ldots + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{h^n}{n!} f^n(a+\theta h)$$

Proof : Let us define a new function ϕ with domain [a, a + h] involving the derivatives of f as follows:

$$\phi(x) = f(x) + (a+h-x) f'(x) + \frac{(a+h-x)^2}{2!} f''(x) + \ldots\ldots\ldots\ldots$$

$$+ \frac{(a+h-x)^{n-1}}{(n-1)!} f^{n-1}(x) + \frac{(a+h-x)^n}{n!} A \qquad \ldots(1)$$

where A is a constant to be determined such that

$$\phi(a+h) = \phi(a) \qquad \ldots(2)$$

From (1), $\phi(a+h) = f(a+h)$ ...(3)

and $\phi(a) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{h^n}{n!} A$ ...(4)

Using (3) and (4) in (2), we obtain

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \ldots + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{h^n}{n!} A \qquad \ldots(5)$$

The value of A is given by (5).

Now by first condition, f(x), f'(x), f''(x)...$f^{n-1}(x)$ are continuous in [a, a + h] and their derivatives exist and are finite in (a, a + h).

Polynomials of x, (a + h − x), $\frac{(a+h-x)^2}{2!}, \ldots, \frac{(a+h-x)^n}{n!}$ are continuous and derivable for all values of x.

Therefore the function is also continuous in [a, a + h] and derivable in (a, a + h).

Thus the function ϕ is :

R_1. Continuous in [a, a + h]

R_2. Derivable in (a, a + h) and

R_3. $\phi(a) = \phi(a+h)$ [by (2)]

Therefore the function ϕ satisfies all the three conditions of Rolle's theorem and accordingly, there exists at least one positive number θ, $0 < \theta < 1$,

where

$$\phi'(a+\theta h) = 0, \qquad 0 < \theta < 1 \qquad \ldots(6)$$

Differentiating (1),

$$\phi'(x) = f'(x) + [(a+h-x) f''(x) - f'(x)]$$

$$+\left[\frac{(a+h-x)^2}{2!}f'''(x)-(a+h-x)f''(x)\right]$$

$$+ \quad \dots \quad \dots \quad \dots \quad \dots \quad +$$

$$+\left[\frac{(a+h-x)^{n-1}}{(n-1)!}f^n(x)-\frac{(a+h-x)}{(n-2)!}f^{n-1}(x)\right]-\frac{(a+h-x)^{n-1}}{(n-1)!}A$$

The terms cancel in pairs on RHS, therefore

$$\phi'(x)=\frac{(a+h-x)^{n-1}}{(n-1)!}f^n(x)-\frac{(a+h-x)^{n-1}}{(n-1)!}A$$

Therefore $\phi'(a+\theta h)=\dfrac{(a+h-a-\theta h)^{n-1}}{(n-1)!}[f''(a+\theta h)-A]$

or, $\quad 0=\dfrac{h^{n-1}(1-\theta)^{n-1}}{(n-1)!}[f''(a+\theta h)-A],$ [by (6)]

$$\phi'(x)=\frac{(a+h-x)^{n-1}}{(n-1)!}f^n(x)-\frac{(a+h-x)^{n-1}}{(n-1)!}A$$

therefore $\phi'(a+\theta h)=\dfrac{(a+h-a-\theta h)^{n-1}}{(n-1)!}[f''(a+\theta h)-A]$

or, $\quad 0=\dfrac{h^{n-1}(1-\theta)^{n-1}}{(n-1)!}[f''(a+\theta h)-A],$ [by(6)]

$\Rightarrow \quad A = f'(a + \theta h) \qquad [\because h \neq 0, (1-\theta) \neq 0] \qquad ...(7)$

Substituting the value of A from (7) in (5), we obtain the required result.

Hence Proved.

Remainder after n terms $[R_n]$:

$$\frac{h^n}{n!}f^n(a+\theta h), 0<\theta<1,$$

This is also known as Lagrange's form of Remainder.

Taylor's theorem with Cauchy's form of remainder :

Statement: If a function f with domain [a, a + h] is such that

(i) f^{n-1} is continuous in [a, a + h]

(ii) f^n exist in (a, a + h)

then $\exists\ \theta \in (0, 1)$ such that :

$$f(a+h)=f(a)+hf'(a)+\frac{h^2}{2!}f''(a)+\dots+\frac{h^{n-1}}{(n-1)!}f^{n-1}(a)+\frac{h^n}{(n-1)}(1-\theta)^{n-1}f^n(a+\theta h).$$

Proof. Consider a function ϕ defined as follows :

$$\phi(x) = f(x) + (a + h - x) f'(x) + \frac{(a+h-x)^2}{2!} f''(x) + \dots$$

$$+\frac{(a+h-x)^{n-1}}{(n-1)!}f^{n-1}(x)+(a+h-x)A \quad ...(1)$$

where A is a constant to be determined such that

$$\phi(a + h) = \phi(a) \quad ...(2)$$

From (1), $\phi(a + h) = f(a + h)$ ...(3)

and $\phi(a) = f(a) + hf^{-1}(a) + ... + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + hA$ ...(4)

Using (3) and (4) in (2),

$$f(a + h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + ... + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + hA \quad ...(5)$$

The value of A is given by (5).

Now by the first condition.

$f(x), f'(x), f''(x), ..., f^{n-1}(x)$ are continuous in [a, a + h] and their derivable $f'(x), f''(x), ..., f^n(x)$ exist and are finite in (a, a + h).

and the polynomials $(a + h - x), \frac{(a+h-x)^2}{2!}, ..., \frac{(a+h-x)^{n-1}}{(n-1)!}$

are continuous and derivable for all values of x. Therefore these are continuous in [a, a + h] and derivable in (a, a + h).

Thus the function ϕ is : R_1. continuous in [a, a + h]

R_2, derivable in (a, a + h)

and R_3, $\phi(a) = \phi(b)$

Thus the function ϕ satisfies all the three conditions of Rolle's theorem, accordingly there exists atleast one positive number θ, $0 < \theta < 1$, where

$$\phi'(a + \theta h) = 0, \quad 0 < \theta < 1 \quad ...(6)$$

Differentiating (1), we get

$$\phi'(x) = f'(x) + [(a + h - x) f''(x) - f'(x)]$$

$$+\left[\frac{(a+h-x)^2}{2!} f'''(x) - (a+h-x) f''(x)\right]$$

$$+ \quad ... \quad ... \quad ... \quad .... \quad +$$

$$+\left[\frac{(a+h-x)^{n-1}}{(n-1)!} f^n(x) - \frac{(a+h-x)^{n-2}}{(n-2)!} f^{n-1}(x)\right] - A$$

The terms cancel in pairs on RHS, therefore

$$\phi'(x) = \frac{(a+h-x)^{n-1}}{(n-1)!} f^n(x) - A$$

Therefore $\phi'(a + \theta h) = \frac{[a+h-(a+\theta h)^{n-1}]}{(n-1)!} f'(a + \theta h) - A$

$$0 = \frac{(h-\theta h)^{n-1}}{(n-1)!} f^n(a+\theta h) - A \quad \text{[by (6)]}$$

$$\Rightarrow \quad A = \frac{h^{n-1}(1-\theta)^{n-1}}{(n-1)!} f^n(a+\theta h) \quad ...(7)$$

Substituting the value of A from (7) in (5), we obtain the required result

Another form :

Replacing h = b – a in the theorem, we get the following useful form :

$$f(b) = f(a) + (b-a) f'(a) + \frac{(b-a)^2}{2!} f''(a) + \ldots$$

$$+ \frac{(b-a)^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{(b-a)^n (1-\theta)^{n-1}}{(n-1)!} f^n[a+\theta(b-a)], 0<\theta<1$$

$$f(a+h) = f(a) + (h) f'(a) + \frac{(h)^2}{2!} f''(a) + \ldots$$

$$+ \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + R_n, 0<\theta<1$$

Remainder after n terms $[R_n]$:

$$R_n = \frac{h^n (1-\theta)^{n-1}}{(n-1)!} f^n(a+\theta h)$$

This is known as Cauchy's form of Remainder.

Taylor's theorem with Schlomitch and Roche form of Remainder.

Statement : If a function f defined in [a, a + h] is such that :

(i) its all derivative upto order (n –1) i.e. f^{n-1} are continuous in [a, a + h]

(ii) f^n exists in (a, a + h)

(iii) $p \in \mathbb{N}$

then $\exists\ \theta \in (0, 1)$ such that :

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \ldots + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{h^n (1-\theta)^{n-p}}{p(n-1)!} f^n(a+\theta h), 0<\theta<1$$

Proof. Consider a function ϕ defined as follows :

$$\phi(x) = f(x) + (a+h-x) f'(x) + \frac{(a+h-x)^2}{2!} f''(x) + \ldots +$$

$$+ \frac{(a+h-x)^{n-1}}{(n-1)!} f^{n-1}(x) + (a+h-x)^p A \quad \ldots(1)$$

where A is a constant to be determined such that

$$\phi(a+h) = \phi(a) \quad \ldots(2)$$

From (1), $\phi(a+h) = f(a+h) \quad \ldots(3)$

and $\phi(a) = f(a) + hf'(a) + \dfrac{h^2}{2!} f''(a) + \ldots$

$$\ldots + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + h^p A \quad \ldots(4)$$

Using (3) and (4) in (2), we get

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \ldots + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + h^p A \quad \ldots(5)$$

The value of A is given by (5).

Now the function ϕ is :

(R_1) continuous in [a, a + h], because

$f, f', f'', ..., f^{n-1}, (a - h - x)^p$ are continuous.

(R_2) derivable in (a, a + h), because

$f, f', f'', ..., f^{n-1}, (a - h - x)^p$ are derivable.

(R_3) $\phi(a + h) = \phi(a)$

"Thus the function f satisfies all the three conditions of Rolle's theorem, according $\exists\theta \in (0, 1)$, where

$$\phi'(a + \theta h) = 0 \quad ...(6)$$

But $$\phi'(x) = \frac{(a+h-x)^{n-1}}{(n-1)!} f^n(x) - Ap(a + h - x)^{p-1}$$

$$\Rightarrow \quad \phi'(a + \theta h) = \frac{h^{n-1}(1-\theta)^{n-1}}{(n-1)!} f^n(a + \theta h) - A\, ph^{p-1} (1 - \theta)^{p-1}$$

$$\Rightarrow \quad \phi'(a + \theta h) = 0$$

$$\Rightarrow \quad A = \frac{(1-\theta)^{n-p}}{p(n-1)!} f^n(a + \theta h) \quad ...(7)$$

Substituting the value of A from (7) in (5), we obtain the required result

Important Special Case :

1. When p = n, then $R_n(x) = \frac{h^n}{n!} f^n(a + \theta h)$,

   which is called Lagrange's form of remainder and we get Taylor's theorem with Lagrange's form of remainder.

2. When p = 1, then $R_n(x) = \frac{h^n(1-\theta)^{n-1}}{(n-1)!} f^n(a + \theta h)$,

   which is called Cauchy's form of remainder and we get Taylor's theorem with Cauchy's form of remainder.

Another form :

Replacing a + h = x or h = x – a, we get the following form of Taylor's theorem :

$$f(x) = f(a) + (x - a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + ... + \frac{(x-a)^{n-1}}{(n-1)!} f^{n-1}(a) + R_n.$$

where $$R_n = \frac{(x-a)^n}{p(n-1)!}(1-\theta)^{n-p} f^n[a + \theta(x-a)] \quad ...(8)$$

is called the remainder after n terms.

Maclaurin's theorem : Statement :

If a function f with domain [0, x] be such that

(1) all of its derivatives upto order (n –1) are continuous in [0, x]

(2) f^n exists in (0, x)

(3) $p \in \mathbb{N}$, Then $\exists\ \theta \in (0, 1)$ such that

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \ldots + \frac{x^{n-1}}{(n-1)!}f^{n-1}(0) + R_n$$

where $R_n = \dfrac{x^n(1-\theta)^{n-p}}{p(n-1)!}f^n(\theta x),\ 0 < \theta < 1.$

Proof. First of all we observe that condition (i) in the statement of theorem implies that f, f', f", ..., f^{n-1} are all defined (i.e., exist) and continuous on [0, x].

Consider the function ϕ defined on [0, x] as

$$\phi(t) = f(t) + (x-t)f'(t) + \frac{(x-t)^2}{2!}f''(t) + \ldots + \frac{(x-t)^{n-1}}{(n-1)!}f^{n-1}(t) + A(x-t)^p$$

where A is a constant to be determined such that $\phi(0) = \phi(x)$.

But $\phi(0) = f(0) + xf'(0) + \dfrac{x^2}{2!}f''(0) + \ldots + \dfrac{x^{n-1}}{(n-1)!}f^{n-1}(0) + Ax^p$

and $\phi(x) = f(x)$

$\therefore$ $\phi(0) = \phi(x)$

$$\Rightarrow \quad f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \ldots + \frac{x^{n-1}}{(n-1)!}f^{n-1}(0) + Ax^p \quad \ldots(1)$$

Now, (i) Since f, f', f", ..., f^{n-1} are all continuous on [0, x] and $(x-t)^r$, $r \in N$ is continuous on R.

$\therefore$ ϕ is continuous on [0, x].

(ii) Since f, f', f", ..., f^{n-1} are all derivable on (0, x) and $(x-t)^r$, $r \in N$ is derivable on R.

$\therefore$ ϕ is derivable on (0, x).

(iii) Also $\phi(0) = \phi(x)$

Thus the function ϕ satisfies all the three conditions of Rolle's Theorem on [0, x] and, hence, there exists a real number $\theta \in (0, 1)$ such that $\phi'(\theta x) = 0$.

But $\phi'(t) = f'(t) - f'(t) + (x-t)f''(t) - (x-t)f''(t) + \ldots$

$$+ \frac{(x-t)^{n-1}}{(n-1)!}f^n(t) - pA(x-t)^{p-1}$$

$$= \frac{(x-t)^{n-1}}{(n-1)!}f^n(t) - pA(x-t)^{p-1} \qquad \text{[other terms cancel in pairs]}$$

$$\Rightarrow \quad \phi'(\theta x) = \frac{(x-\theta x)^{n-1}}{(n-1)!}f^n(\theta x) - pA(x-\theta x)^{p-1}$$

$$= \frac{x^{n-1}(1-\theta)^{n-1}}{(n-1)!}f^n(\theta x) - pAx^{p-1}(1-\theta)^{p-1}$$

$\therefore$ $\phi'(\theta x) = 0$

$$\Rightarrow \quad \frac{x^{n-1}(1-\theta)^{n-1}}{(n-1)!}f^n(\theta x) = pAx^{p-1}(1-\theta)^{p-1}$$

$$\Rightarrow \qquad A = \frac{x^{x-p}(1-\theta)^{n-p}}{p(n-1)!} f^n(\theta x)$$

Putting this value of A in (1), we have

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + ... + \frac{x^{n-1}}{(n-1)!} f^{n-1}(0) + \frac{x^n(1-\theta)^{n-p}}{p(n-1)} f^n(\theta x)$$

The term $\quad R_n = \dfrac{x^n (1-\theta)^{n-p}}{p(n-1)} f^n(\theta x)$

which occurs after n terms in known as **Schlomilch and Roche's form of remainder.**

Note. (i) For p = 1, we get $R_n = \dfrac{x^n(1-\theta)^{n-1}}{(n-1)!} f^n(\theta x)$ called **Cauchy's form of remainder.**

(ii) For p = n, we get $R_n = \dfrac{x^n}{n!} f^n(\theta x)$ called **Lagrange's form of remainder.**

Ex : Prove that the number 'θ' which occurs in the Taylor's theorem with Lagrange's form of remainder after n terms approaches the limit $\dfrac{1}{n+1}$ as h approaches zero provided that $f^{n+1}(x)$ is continuous and different from zero at x = a.

Sol. By Taylor's theorem with Lagrange's form of remainder after n terms and (n + 1) terms successively, we have

$$f(a + h) = f(a) + hf'(a) + ... + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{h^n}{n!} f^n(a + \theta h) \quad \text{where } 0 < \theta < 1.$$

$$f(a + h) = f(a) + hf'(a) + ... + \frac{h^n}{n!} f^n(a) + \frac{h^{n+1}}{(n+1)!} f^{n+1}(a + \theta_1 h) \text{ where } 0 < \theta_1 < 1.$$

On subtraction, we get $0 = \dfrac{h^n}{n!} [f^n(a + \theta h) - f^n(a)] - \dfrac{h^{n+1}}{(n+1)!} f^{n+1}(a + \theta_1 h)$

or $\qquad f^n(a + \theta h) - f^n(a) = \dfrac{h}{n+1} f^{n+1}(a + \theta_1 h) \qquad$ ...(1)

Using Lagrange's mean value theorem for $f^n(x)$ on [a, a + θh], we have

$$f^n(a + \theta h) = f^n(a) + \theta h f^{n+1}(a + \theta_2 \theta h) \text{ where } 0 < \theta_2 < 1$$

or $\qquad f^n(a + \theta h) - f^n(a) = \theta h f^{n+1}(a + \theta_2 \theta h) \qquad$ ...(2)

From (1) and (2), we have

$$\theta h f^{n+1}(a + \theta_2 \theta h) = \frac{h}{n+1} f^{n+1}(a + \theta_1 h) \text{ or } \theta = \frac{1}{n+1} \cdot \frac{f^{n+1}(a + \theta_1 h)}{f^{n+1}(a + \theta_2 \theta h)}$$

$$\therefore \quad \lim_{h \to 0} \theta = \lim_{h \to 0} \frac{1}{n+1} \cdot \frac{f^{n+1}(a + \theta_1 h)}{f^{n+1}(a + \theta_2 \theta h)} = \frac{1}{n+1} \cdot \frac{f^{n+1}(a)}{f^{n+1}(a)} \quad [\because f^{n+1}(x) \text{ is continuous}]$$

$$= \frac{1}{n+1}. \qquad [\because f^{n+1}(a) \neq 0]$$

Ex : Assuming the derivatives which occur are continuous, apply the mean value theorem to prove that

$$\phi'(x) = F'\{f(x)\}\, f'(x) \text{ where } \phi(x) = F\{f(x)\}.$$

Sol. Let $f(x) = t$ so that $\phi(x) = F(t)$

Now $\phi'(x) = \lim_{h\to 0} \dfrac{\phi(x+h)-\phi(x)}{h} = \lim_{h\to 0} \dfrac{F\{f(x+h)\}-F\{f(x)\}}{h}$

$$= \lim_{h\to 0} \frac{F\{f(x)+hf'(x+\theta_1 h)\}-F\{f(x)\}}{h} \text{ where } \theta < \theta_1 < 1$$

[$\because$ $f(x + h) = f(x) + hf'(x + \theta_1 h)$ by Mean Value Theorem]

$$= \lim_{h\to 0} \frac{F(t+H)-F(t)}{h} \text{ where } H = hf'(x+\theta_1 h)$$

$$= \lim_{h\to 0} \frac{F(t)+HF'(t+\theta_2 H)-F(t)}{hh} \text{ where } 0 < \theta_2 < 1$$

[$\because$ $F(t + H) = F(t) + HF'(t + \theta_2 H)$ by Mean Value Theorem]

$$= \lim_{h\to 0} \frac{HF'(t+\theta_2 H)}{h} = \lim_{h\to 0} \frac{hf'(x+\theta_1 h)F\{t+\theta_2 hf'(x+\theta_1 h)\}}{h}$$

$$= \lim_{h\to 0} f'(x+\theta_1 h)\, F'\{f(x)+\theta_2 hf'(x+\theta_1 h)\}$$

$= f'(x)\, F'(f(x))$ [$\because$ f' and F' are continuous]

$= F'\{f(x)\}\, f'(x).$

Ex : Using Taylor's theorem, show that

(i) $\cos x \ge 1 - \dfrac{x^2}{2}\ \forall\, x \in R$

(ii) $1 + x + \dfrac{x^2}{2} < e^x < 1 + x + \dfrac{x^2}{2} e^x,\ x > 0$

(iii) $x - \dfrac{x^3}{3!} < \sin x < x,\ x > 0$

(iv) $x - \dfrac{x^3}{3!} \le \sin x \le x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!},\ x \ge 0$

Sol. (i) **Case 1.** Let $x = 0$

The $\cos x = 1,\ 1 - \dfrac{x^2}{2} = 1$ $\therefore$ $\cos x = 1 - \dfrac{x^2}{2}$.

Case 2. Let $x > 0$ and $f(x) = \cos x$

Then $f'(x) = -\sin x,\ f''(x) = -\cos x$

Since $f(x) = f(0) + xf'(0) + \dfrac{x^2}{2!} f''(\theta x)$ where $0 < \theta < 1$.

$\therefore$ $\cos x = 1 - \dfrac{x^2}{2} \cos \theta x$

But $\cos \theta < 1\ \theta x,\ x > 0$

$\therefore$ $1 - \dfrac{x^2}{2} \cos \theta x > 1 - \dfrac{x^2}{2}$ $\Rightarrow$ $\cos x > 1 - \dfrac{x^2}{2}$.

Case 3. Let $x < 0$

Put $y = -x$ so that $y > 0$

By case 2, $\cos y > 1 - \frac{y^2}{2} \Rightarrow \cos(-x) > 1 - \frac{(-x)^2}{2} \Rightarrow \cos x > 1 - \frac{x^2}{2}$

Combining all cases, $\cos x \geq 1 - \frac{x^2}{2} \quad \forall x \in R$.

(ii) Let $f(x) = e^x, e > 0$, then $f'(x) = f''(x) = e^x$

Since $f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(\theta x)$ where $0 < \theta < 1$

$\therefore \quad e^x = 1 + x + \frac{x^2}{2} e^{\theta x}$

Now $0 < \theta < 1$ and $x > 0 \Rightarrow 0 < \theta x < x$

$\Rightarrow \quad e^0 < e^{\theta x} < e^x$ [$\because$ e^x is an increasing function]

$$\Rightarrow \frac{x^2}{2} < \frac{x^2}{2} e^{\theta x} < \frac{x^2}{2} e^x \Rightarrow 1 + x + \frac{x^2}{2} < 1 + x + \frac{x^2}{2} e^{\theta x} < 1 + x + \frac{x^2}{2} e^x$$

$$\Rightarrow 1 + x + \frac{x^2}{2} < e^x < 1 + x + \frac{x^2}{2} e^x.$$

(iii) Let $f(x) = \sin x, x > 0$, then $f'(x) = \cos x, f''(x) = -\sin x$

$f'''(x) = -\cos x$

Since $f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(\theta x)$ where $0 < \theta < 1$

$\therefore \quad \sin x = x - \frac{x^3}{3!} \cos \theta x$

But $\cos \theta x < 1 \quad \forall \theta x, x > 0$

$\therefore \quad x - \frac{x^3}{3!} \cos \theta x > x - \frac{x^3}{3!}$

$\Rightarrow \quad \sin x > x - \frac{x^3}{3!}$ ...(1)

Also $f(x) = f(0) + xf'(\theta x)$ where $0 < \theta < 1$

$\therefore \quad \sin x = x \cos \theta x$

But $\cos \theta x < 1 \quad \forall \theta x, x > 0$

$\therefore \quad x \cos \theta x < x \Rightarrow \sin x < x$ ...(2)

Combining (1) and (2), we get $x - \frac{x^3}{3!} < \sin x < x$.

(iv) **Case 1.** Let $x = 0$

Then $x - \frac{x^3}{3!} = \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!}$ (each $x = 0$).

Case 2. Let $x > 0$ and $f(x) = \sin x$

Then $f'(x) = \cos x, f''(x) = -\sin x, f'''(x) = -\cos x,$

$f^{iv}(x) = \sin x, f^{v}(x) = \cos x$

Since $\quad f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(\theta x)$

$\therefore \quad \sin x = x - \frac{x^3}{3!} \cos \theta x$

But $\quad \cos \theta x < 1 \ \forall \ \theta x, x > 0$

$\therefore \quad x - \frac{x^3}{3!} \cos \theta x > x - \frac{x^3}{3!}$

$\Rightarrow \quad \sin x > x - \frac{x^3}{3!}$ ...(1)

Also $\quad f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{iv}(0) + \frac{x^5}{5!} f^{v}(\theta x)$

$\therefore \quad \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} \cos \theta x$

But $\quad \cos \theta x < 1 \ \forall \ \theta x, x > 0$

$\therefore \quad x - \frac{x^3}{3!} + \frac{x^5}{5!} \cos \theta x < x - \frac{x^3}{3!} + \frac{x^5}{5!}$

$\Rightarrow \quad \sin x < x - \frac{x^3}{3!} + \frac{x^5}{5!}$ ...(2)

Combining (1) and (2), $\quad x - \frac{x^3}{3!} < \sin x < x - \frac{x^3}{3!} + \frac{x^5}{5!}$

Combining the two cases, we have $x - \frac{x^3}{3!} \le \sin x \le x - \frac{x^3}{3!} + \frac{x^5}{5!}$, $x > 0$.

Taylor's Series :

If a function f is defined in the interval [a + ah] and satisfy all conditions of Taylor's theorem, then by Taylor's theorem,

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + ... + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + R_n$$

or, $\quad f(a + h) = S_n + R_n$, where S_n represents the sum of first n terms and R_n is called the Taylor's remainder after n terms.

$$\lim_{n\to\infty} f(a+h) = \lim_{n\to\infty} S_n + \lim_{n\to\infty} R_n$$

or, $\quad f(a + h) = \lim_{n\to\infty} S_n$

Therefore if the function f is expressible in an ***Infinite series***, then clearly

(1) $f^n(a)$ exists $\forall \ n \in \mathbf{N}$ and

(2) for the convergence of the Taylor's series,
Taylor's remainder after n terms, $R_n \to 0$ when $n \to \infty$, then

$$f(a + h) = f(a) + h f'(a) + \frac{h^2}{2!} f''(a) + ... + \frac{h^n}{n!} f^n(a) + ...$$

The infinite series on RHS is known as **Taylor's Series.**

Maclaurin's Series :

When a = 0 and replace h by x, then the Taylor's series reduces to the following form :

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \ldots + \frac{x^n}{n!} f^n(0) + \ldots$$

This series is called Maclaurin's Series and is very useful in the expansion of functions.

Ex : Expand e^x as an infinite series.

Sol. Let $f(x) = e^x$ so that $f^n(x) = e^x$ and $f^n(0) = e^0 = 1 \ \forall \ n \in N$

Clearly, f and all its derivatives exist and are continuous for every real value of x.

Lagrange's form of remainder is

$$R_n = \frac{x^n}{n!} f^n(\theta x),\ 0 < \theta < 1 = \frac{x^n}{n!} \cdot e^{\theta x}$$

Let $\quad a_n = \frac{x^n}{n!} \ \forall \ n \in N$ then $a_{n+1} = \frac{x^{(n+1)}}{(n+1)!}$

so that $\quad \lim\limits_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim\limits_{n \to \infty} \frac{x}{n+1} = 0 < 1$

$\Rightarrow \quad \lim\limits_{n \to \infty} a_n = 0 \qquad \left[\because \text{ if } \lim\limits_{n \to \infty} \frac{a_{n+1}}{a_n} = l \text{ and } |l| < 1 \text{ then } \lim\limits_{n \to \infty} a_n = 0\right]$

$\Rightarrow \quad \lim\limits_{n \to \infty} \frac{x^n}{n!} = 0$

$\therefore \quad \lim\limits_{n \to \infty} R_n = \lim\limits_{n \to \infty} \frac{x^n}{n!} e^{\theta x} = \left(\lim\limits_{n \to \infty} \frac{x^n}{n!}\right) e^{\theta x} = 0 \times e^{\theta x} = 0$

Thus the conditions of Maclaurin's infinite expansion are satisfied.

$\therefore$ For all $x \in R$, $\quad e^x = f(x)$

$$= f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \ldots + \frac{x^n}{n!} f^n(0) + \ldots$$

$$= 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Ex : Expand sin x as an infinite series.

Sol. Let $f(x) = \sin x$ so that $f^n(x) = \sin\left(\frac{n\pi}{2} + x\right)$

and $\quad f^n(0) = \sin \frac{n\pi}{2} \ \forall \ n \in N$

$$\Rightarrow \quad f^n(0) = \begin{cases} 0 & , \text{ if n is even} \\ (-1)^{\frac{n-1}{2}}, & \text{ if n is odd} \end{cases}$$

$\Rightarrow \quad f''(0) = f^{iv}(0) = \ldots = 0,\ f'(0) = 1,\ f'''(0) = -1,\ f^v(0) = 1, \ldots$

Clearly, f and all its derivatives exist and are continuous for every real value of x.

Lagrange's form of remainder is

$$R_n = \frac{x^n}{n!} f^n(\theta x), 0 < \theta < 1 = \frac{x^n}{n!} \sin\left(\frac{n\pi}{2} + \theta x\right)$$

$$\therefore \quad | R_n | = \left|\frac{x^n}{n!} \sin\left(\frac{n\pi}{2} + \theta x\right)\right|$$

$$= \left|\frac{x^n}{n!}\right| \left|\sin\left(\frac{n\pi}{2} + \theta x\right)\right| \le \left|\frac{x^n}{n!}\right| \qquad \left[\because \left|\sin\left(\frac{n\pi}{2} + \theta x\right)\right| \le 1\right]$$

But $\lim_{n\to\infty} \left|\frac{x^n}{n!}\right| = 0 \ \forall \ x \in R$

$\therefore \quad \lim_{n\to\infty} | R_n | = 0$ and hence $\lim_{n\to\infty} R_n = 0$

Thus the conditions of Maclaurin's infinite expansion are satisfied.

$\therefore \quad$ For all $x \in R$, $\sin x = f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \ldots = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$

Ex : Expand $(1 + x)^m$, $m \in R$.

Sol. Two cases arise according as m is not a positive integer.

Case 1. When m is a positive integer.

Let $\quad f(x) = (1 + x)^m, x \in R$

Then $f^n(x)$ exists for all x and all n.

In fact, if $1 \le n \le m$, then $\quad f^n(x) = m(m - 1)(m - 2) \ldots (m - n + 1) (1 + x)^{m-n}$

so that $\quad f^m(x) = m!$ and $f^n(x) = 0$, if $n > m$

$$\Rightarrow \quad f^n(0) = \begin{cases} m(m-1)\ldots(m-n+1) & \text{if } 1 \le n \le m \\ 0 & \text{if } n > m \end{cases}$$

Since $f^n(x) = 0$ for all $n > m$, it follows that $R_n \to 0$ as $n \to \infty$. Thus the conditions of Maclaurin's expansion are satisfied.

$$\therefore \quad (1 + x)^m = f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \ldots + \frac{x^m}{m!} f^m(0) + 0 + 0 + \ldots$$

$$= 1 + mx + \frac{m(m-1)}{2!} x^2 + \ldots + x^m.$$

Case 2. When m is not a positive integer.

Let $\quad f(x) = (1 + x)^m, x \ne -1$

Taking Cauchy form of remainder, we have

$$R_n = \frac{x^n}{(n-1)} (1 - \theta)^{n-1} f^n(\theta x), \ 0 < \theta < 1$$

$$= \frac{x^n}{(n-1)!} (1 - \theta)^{n-1}. \ m(m - 1) \ldots (m - n + 1)(1 + \theta x)^{m-n}$$

$$= \left(\frac{m(m-1)...(m-n+1)}{(n-1)!}x^n\right)\left(\frac{1-\theta}{1+\theta x}\right)^{n-1}(1+\theta x)^{m-1}$$

Now let $\quad a_n = \dfrac{m(m-1)...(m-n+1)}{(n-1)!}x^n$

so that $\quad a_{n+1} = \dfrac{m(m-1)...(m-n+1)(m-n)}{n!}x^{n+1}$

$\therefore \quad \dfrac{a_{n+1}}{a_n} = \left(\dfrac{m-n}{n}\right)x = \left(\dfrac{m}{n}-1\right)x$

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\left(\frac{m}{n}-1\right)x = -x$$

It follows that if $|-x| = |x| < 1$, then $\lim\limits_{n\to\infty} a_n = 0$

$\Rightarrow \quad \dfrac{m(m-1)...(m-n+1)}{(n-1)!}x^n \to 0$ for $|x| < 1$ ...(1)

Since $\quad 0 < \theta < 1 \quad$ and $\quad -1 < x < 1$

$\therefore \quad -\theta < \theta x < \theta \quad \Rightarrow \quad 1 - \theta < 1 + \theta x < 1 + \theta$

$\Rightarrow \quad 0 < 1 - \theta < 1 + \theta x \quad \Rightarrow \quad 0 < \dfrac{1-\theta}{1+\theta x} < 1$

Consequently, $\quad \lim\limits_{n\to\infty}\left(\dfrac{1-\theta}{1+\theta x}\right)^{n-1} = 0$ ...(2)

Also, since $-|x| \le x \le |x|$ and $0 < \theta < 1 \Rightarrow -\theta|x| \le \theta x \le \theta|x|$

$\Rightarrow \quad -|x| < -\theta|x| \le \theta x \le \theta|x| < |x| \quad \Rightarrow \quad 1 - |x| < 1 + \theta x < 1 + |x|$

$$\left.\begin{aligned} &\therefore \text{ If } m > 1, \text{ then } (1+\theta x)^{m-1} < (1+|x|)^{m-1} \\ &\text{and if } m < 1, \text{ then } (1+\theta x)^{m-1} = \frac{1}{(1+\theta x)^{1-m}} \\ &< \frac{1}{(1-|x|)^{1-m}} = (1-|x|)^{m-1} \end{aligned}\right\} \quad ...(3)$$

From (1), (2) and (3), we find that for $|x| < 1$, $\lim\limits_{n\to\infty} R_n = 0$.

Thus the conditions of Maclaurin's infinite expansion are satisfied and

$$(1+x)^m = f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + ... + \frac{x^n}{n!}f^n(0) + ...$$

$$= 1 + mx + \frac{m(m-1)}{2!}x^2 + ... + \frac{m(m-1)....(m-n+1)}{n!}x^n + ...$$

Ex : Expand log (1 + x) as an infinite series.

Sol. Let $\quad f(x) = \log(1+x)$ where $1 + x > 0$ i.e., $x > -1$

Then $f^n(x) = \dfrac{(-1)^{n-1}(n-1)!}{(1+x)^n} \forall\ n \in \mathbf{N}$ and $x > -1$.

Case 1. When $0 \le x \le 1$

Writing Lagrange's remainder after n terms, we have

$$R_n = \frac{x^n}{n!} f^n(\theta x),\ 0 < \theta < 1$$

$$= \frac{x^n}{n!} \cdot \frac{(-1)^{n-1}(n-1)!}{(1+\theta x)} = \frac{(-1)^{n-1}}{n} \cdot \left(\frac{x}{1+\theta x}\right)^n$$

If $x = 1$, then $| R_n | = \dfrac{1}{n}\left(\dfrac{1}{1+\theta}\right)^n < \dfrac{1}{n}$ and $\dfrac{1}{n} \to 0$ as $n \to \infty$.

$\therefore \quad \lim_{n\to\infty} R_n = 0$

If $\quad 0 \le x < 1$, then since $0 < \theta < 1$

$\therefore \quad 0 \le x < 1 + \theta x$

$\Rightarrow \quad 0 \le \dfrac{x}{1+\theta x} < 1 \quad \Rightarrow \quad 0 \le \left(\dfrac{x}{1+\theta x}\right)^n < 1$

$\therefore \quad | R_n | = \dfrac{1}{n}\left(\dfrac{x}{1+\theta x}\right)^n < \dfrac{1}{n}$ and $\dfrac{1}{n} \to 0$ as $n \to \infty$.

$\therefore \quad \lim_{n\to\infty} R_n = 0.$

Case 2. When $-1 < x < 0$

Since in this case, $\left|\dfrac{x}{1+\theta x}\right|$ need not be less than unity, therefore, it may not be easily shown that $R_n \to 0$ as $n \to \infty$ by considering Lagrange's remainder.

$\therefore \quad$ Writing Cauchy's remainder after n terms, we have

$$R_n = \frac{x^n}{(n-1)!}(1-\theta)^{n-1} f^n(\theta x),\ 0 < \theta < 1$$

$$= \frac{x^n}{(n-1)!}(1-\theta)^{n-1} \cdot \frac{(-1)^{n-1}(n-1)!}{(1+\theta x)^n} = (-1)^{n-1} \cdot x^n \left(\frac{1-\theta}{1+\theta x}\right)^{n-1} \cdot \frac{1}{1+\theta x}$$

Now $\quad -1 < x < 0$ and $0 < \theta < 1 \Rightarrow -\theta < \theta x$

$\Rightarrow \quad 1 - \theta < 1 + \theta x \quad \Rightarrow \quad 0 < \dfrac{1-\theta}{1+\theta x} < 1$

Also, $\quad -| x | \le x \quad \Rightarrow \quad -\theta | x | \le \theta x$

$\therefore \quad -| x | < -\theta | x | \le \theta x$

$\Rightarrow \quad 1 - | x | < 1 + \theta x \quad \Rightarrow \quad \dfrac{1}{1+\theta x} < \dfrac{1}{1-| x |}$

Consequently, $|R_n| = |x|^n \left|\frac{1-\theta}{1+\theta x}\right|^{n-1} \left|\frac{1}{1+\theta x}\right| < \frac{|x|^n}{1-|x|}$

and $|x|^n \to 0$ as $n \to \infty$ (since $|x| < 1$)

$\therefore \quad \lim_{n\to\infty} R_n = 0$

Thus, we find that if $-1 < x \le 1$, then $\lim_{n\to\infty} R_n = 0$.

Hence $\log(1 + x) = f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ldots$

$$= \log 1 + x.1 + \frac{x^2}{2!}(-1) + \frac{x^2}{3!}.2 \ldots = x - \frac{x^2}{2} + \frac{x^3}{3} \ldots$$

4. MAXIMA AND MINIMA OF ONE VARIABLE

Let I be an interval.

A function $f : I \to \mathbb{R}$ is said to have a global maxima (or an absolute maximum) on I if there exists a point $c \in I$ such that $f(c) \geq f(x)$ for all $x \in I$. c is said to be a point of global maxima for f on I.

f is said to have a global minima (or an absolute minimum) on I if there exists a point $c \in I$ such that $f(c) \leq f(x)$ for all $x \in I$. c is said to be a point of global minima for f on I.

A function $f : I \to \mathbb{R}$ is said to have a local maximum (or a relative maximum) at a point $c \in I$ if there exists a neighbourhood $N(c, \delta)$ of c such that $f(c) \geq f(x)$ for all $x \in N(c, \delta) \cap I$.

f is said to have a local minimum (or a relative minimum) at a point $c \in I$, If there exists a neighbourhood $N(c, \delta)$ of c such that $f(c) \leq f(x)$ for all $x \in N(c, \delta) \cap I$.

We say that f has a local extremum (or a relative extremum) at a point $c \in I$, if f has either a local maximum or a local minimum at c.

Note : If $f : I \to \mathbb{R}$ has a local maximum (a local minimum) at a point $c \in I$ then c is a point of global maxima (a point of global minima) for f on $N(c, \delta) \cap I$ for some suitable $\delta > 0$.

$N(c, \delta) = (c - \delta, c + \delta)$

Theorem : Let $f : I \to \mathbb{R}$ be such that f has a local extremum at an interior point c of I. If $f'(c)$ exists then $f'(c) = 0$.

Proof. We prove that theorem for the case when f has a local maximum at c. The proof of the other case is similar.

Since $f'(c)$ exists, either $f'(c) > 0$, or $f'(c) < 0$, or $f'(c) = 0$.

Let $f'(c) > 0$. Then $\lim\limits_{n \to c} \dfrac{f(x) - f(c)}{x - c} > 0$.

Therefore there exists a positive δ such that $\dfrac{f(x) - f(c)}{x - c} > 0$ for all $x \in N'(c, \delta) \cap I$.

Let $c < x < c + \delta$. Then $x - c > 0$ and therefore $f(x) > f(c)$ for all $x \in (c, c + \delta)$. This contradicts that f has a local maximum at c.

Consequently, $f'(c) \geq 0$ ... ... (i)

Let $f'(c) < 0$. Then $\lim\limits_{x \to c} \dfrac{f(x) - f(c)}{x - c} < 0$.

Therefore there exists a positive δ such that $\dfrac{f(x) - f(c)}{x - c} < 0$ for all $x \in N'(c, \delta) \cap I$.

Let $c - \delta < x < c$. Then $x - c < 0$ and therefore $f(x) > f(c)$ for all $x \in (c - \delta, c)$. This contradicts that f has a local maximum at c.

Consequently, $f'(c) \leq 0$ ... ... (ii)

From (i) and (ii) we have $f'(c) = 0$.

This completes the proof.

Corollary : Let $f : I \to \mathbb{R}$ and c be an interior point of I, where f has a local extremum. Then either $f'(c)$ does not exist, or $f'(c) = 0$.

Note 1. The theorem says that if the derivative $f'(c)$ exists at an interior point c of local extremum, $f'(c)$ must be 0. A function may, however have a local extremum at an interior point c of its domain without being differentiable at c. For example, the function defined by $f(x) = |x|$, $x \in R$ has a local minimum at 0 but $f'(0)$ does not exist.

Note 2. The condition $f'(c) = 0$ (when $f'(c)$ exists) is only a necessary condition for an interior point c to be a point of local extremum of the function f.

For example, for the function f defined by $f(x) = x^3$, $x \in R$, 0 is an interior point of the domain of f. $f'(0) = 0$ but 0 is neither a point of local maximum nor a point of local minimum of the function f.

Note 3. The theorem holds if c is an interior point of I.

Let a function f be defined on [0, 1] by $f(x) = x$, $x \in [0, 1]$. Then f has a local maximum at 1 (not an interior point of I), f is differentiable at 1, but $f'(1) \neq 0$.

Theorem : (First derivative test for extrema)

Let f be continuous on $I = [a, b]$ and c be an interior point of I. Let f be differentiable on (a, c) and (c, b).

1. If there exists a neighbourhood $(c - \delta, c + \delta) \subset I$ such that $f'(x) \geq 0$ for $x \in (c - \delta, c)$ and $f'(x) \leq 0$ for $x \in (c, c + \delta)$, then f has a local maximum at c.
2. If there exists a neighbourhood $(c - \delta, c + \delta) \subset I$ such that $f'(x) \leq 0$ for $x \in (c - \delta, c)$ and $f'(x) \geq 0$ for $x \in (c, c + \delta)$, then f has a local minimum at c.
3. If $f'(x)$ keeps the same sign on $(c - \delta.c)$ and $(c.c + \delta)$, then f has no extremum at c.

Proof. 1. Let $x \in (c - \delta, c)$. Applying Mean value theorem to the function f on [x, c], we have $f(c) - f(x) = (c - x) f'(\xi)$ for some $x \in (x, c)$.

Since $f'(\xi) \geq 0$, we have $f(x) \leq f(c)$ for $x \in (c - \delta, c)$.

Let $x \in (c, c + \delta)$. Applying Mean value theorem to the function f on [c, x], we have $f(x) - f(c) = (x - c) f'(\eta)$ for some $\eta \in (c, x)$.

Since $f'(\eta) \leq 0$, we have $f(x) \leq f(c)$ for $x \in (c, c + \delta)$.

If follow that $f(c) \geq f(x)$ for all $x \in N(c, \delta) \cap I$.

Therefore f has a local maximum at c.

2. Similar proof.

3. Let $f'(x) > 0$ for $x \in (c - \delta, c)$ and for $x \in (c, c + \delta)$.

Then $f(x) < f(c)$ for $x \in (c - \delta, c)$ and $f(c) < f(x)$ for $x \in (c, c + \delta)$.

Therefore f has neither a maximum nor a minimum at c.

Similar proof if $f'(x) < 0$ for $x \in (c - \delta, c)$ and for $(c, c + \delta)$.

Note. The converse of the theorem is not true.

For example, let $f(x) = 2x^2 + x^2 \sin \dfrac{1}{x}$, $x \neq 0$

$= 0$, $x = 0$.

Then f has a local minimum at 0.

$f'(x) = 4x + 2x \sin \dfrac{1}{x} - \cos\dfrac{1}{x}$. $x \neq 0$

$= 0$, $x = 0$.

f' takes both positive and negative values on both sides of 0 (in the immediate neighbourhood).

Examples :

1. Let $f(x) = |x|$, $x \in \mathbb{R}$

f is continuous on $\mathbb{R}$. f is not differentiable at 0.

$f'(x) < 0$ for $x \in (-\delta,0)$ and $f'(x) > 0$ for $x \in (0, \delta)$ for some $\delta > 0$.

Therefore f has a local minimum at 0.

2. Let $f(x) = |x - 1| + |x - 2|$, $x \in [0, 3]$

Then $f(x) = 3 - 2x$, if $0 \le x < 1$

$= 1$, if $1 \le x \le 2$

$= 2x - 3$, if $2 < x \le 3$.

f is continuous on [0, 3]. f is not differentiable at 1 and 2.

$f'(x) < 0$ for $x \in (1 - \delta, 1)$, $f'(x) = 0$ for $x \in (1, 1 + \delta)$ for some δ satisfying $0 < \delta < 1$. Therefore f has a local minimum at 1.

$f'(x) = 0$ for $x \in (2 - \delta, 2)$, $f'(x) > 0$ for $x \in (2, 2 + \delta)$ for some δ satisfying $0 < \delta < 1$. Therefore f has a local minimum at 2.

3. $f(x) = (x - 1)^2 (x - 3)^3$, $x \in R$.

$f'(x) = 2(x - 1)(x - 3)^3 + 3(x - 1)^2 (x - 3)^2$

$= (x - 1)(x - 3)^2 (5x - 9)$, $x \in R$.

f is continuous on R. $f'(x) = 0$ at the points 1, 3, $\frac{9}{5}$.

$f'(x) > 0$ for $x \in (1 - \delta, 1)$ and $f'(x) < 0$ for $x \in (1, 1 + \delta)$ for some $\delta > 0$. Therefore f has a local maximum at 1.

$f'(x) > 0$ for $x \in (3 - \delta, 3)$ and $f'(x) > 0$ for $x \in (3, 3 + \delta)$ for some $\delta > 0$. Therefore f has neither a maximum nor a minimum at 3.

$f'(x) < 0$ for $x \in \left(\frac{9}{5} - \delta, \frac{9}{5}\right)$ and $f'(x) > 0$ for $x \in \left(\frac{9}{5}, \frac{9}{5} + \delta\right)$ for some $\delta > 0$. Therefore f has a local minimum at $\frac{9}{5}$.

Theorem : (Higher order derivative test for extrema)

Let $f : I \to R$ and c be an interior point of I.

If $f'(c) = f''(c) = \ldots = f^{n-1}(c) = 0$ and $f^n(c) \neq 0$, then f has

(i) no extremum at c if n be odd, and

(ii) a local extremum at c if n be even:

a local maximum if $f^n(c) < 0$, a local minimum if $f^n(c) > 0$.

Example :

1. $f(x) = x^5 - 5x^4 + 5x^3 + 10$, $x \in R$.

Show that f has a maximum at 1 and a minimum at 3 and f has neither a maximum nor a minimum at 0.

For an extremum $f'(x) = 0$. $f'(x) = 0$ at $x = 1, 3, 0$.

$f''(x) = 20x^3 - 60x^2 + 30x$. Therefore $f''(1) < 0$, $f''(3) > 0$, $f''(0) = 0$.

Since $f'(1) = 0$ and $f''(1) < 0$, f has a local maximum at 1.

Since $f'(3) = 0$ and $f''(3) > 0$, f has a local minimum at 3.

Since $f'(0)$ and $f''(0) = 0$, in order to decide the nature of f at 0, we are to examine derivative of higher order at 0.

$f'''(x) = 60x^2 - 120x + 30$. $f'''(0) = 30 \neq 0$.

Therefore f has neither a maximum nor a minimum at 0.

2. Find the local extremum point of the function $f(x) = \frac{x^2}{(1-x)^3}$, $x \neq 1$.

$$f'(x) = \frac{2(1-x)^3 x + 3x^2(1-x)^2}{(1-x)^6} = \frac{x(1-x)^2(x+2)}{(1-x)^6} = \frac{x(x+2)}{(1-x)^4}.$$

$f'(x) = 0$ at $x = -2, 0$.

Let h be an arbitrarily small positive number.

$f'(-2-h) > 0$, $f'(-2) = 0$, $f'(-2+h) < 0$.

$f'(0-h) < 0$, $f'(0) = 0$, $f'(0+h) > 0$.

f is continuous at -2. $f'(x) > 0$ for $x \in (-2-\delta, 2)$ and $f'(x) < 0$ for $x \in (-2, -2+\delta)$ for some $\delta > 0$.

f is continuous at 0. $f'(x) < 0$ for $x \in (-\delta, 0)$ and $f'(x) > 0$ for $x \in (0, \delta)$ for some $\delta > 0$.

Hence f has a local maximum at -2 and a local minimum at 0.

3. Find the global maximum and the global minimum of the function f on R, where $f(x) = \frac{x^2-2x+4}{x^2+2x+4}$, $x \in R$.

$$f'(x) = \frac{4(x^2-4)}{(x^2+2x+4)^2}.$$

$f'(x) = 0$ at $x = \pm 2$. $f'(x) < 0$ for $|x| < 2$ and $f'(x) > 0$ for $|x| > 2$.

f is continuous at 2. $f'(2+h) > 0$ and $f'(2-h) < 0$ for sufficiently small $h > 0$. Therefore f has a local minimum at 2 and $f(2) = \frac{1}{3}$.

f is continuous at -2. $f'(-2+h) < 0$ and $f'(-2-h) > 0$ for sufficiently small $h > 0$. Therefore f has a local maximum at -2 and $f(-2) = 3$.

As $f'(x) > 0$ for $x > 2$ and f is continuous at 2, f is an increasing function $[2, \infty)$ and $\lim_{n\to\infty} f(x) = 1$.

Therefore $\sup_{x\in[2,\infty)} f(x) = 1$ and $\inf_{x\in[2,\infty)} f(x) = f(2) = \frac{1}{3}$.

As $f'(x) > 0$ for $x < -2$ and f is continuous at -2, f is an increasing function on $(-\infty, -2]$ and $\lim_{x\to-\infty} f(x) = 1$.

Therefore $\sup_{x\in(-\infty,-2]} f(x) = f(-2) = 3$ and $\inf_{x\in(-\infty,-2]} f(x) = 1$.

$\sup_{x\in[-2,2]} f(x) = 3$ and $\inf_{x\in[-2,2]} f(x) = \frac{1}{3}$.

Therefore $\sup_{x\in R} f(x) = f(-2) = 3$ and $\inf_{x\in R} f(x) = f(2) = \frac{1}{3}$.

Ex : Find the maximum and minimum points of the function f given by

$$f(x) = (x-1)(x-2)(x-3).$$

Sol. $f(x) = x^3 - 6x^2 + 11x - 6$

$$f'(x) = 3x^2 - 12x + 11$$

$$f''(x) = 6x - 12$$

For maximum or minimum,

$$f'(x) = 0 \Rightarrow 3x^2 - 12x + 11 = 0$$

$$\Rightarrow \quad x = \frac{12 \pm \sqrt{144+132}}{6} = \frac{12+2\sqrt{3}}{6} = 2 \pm \frac{1}{3}\sqrt{3}$$

At $x = 2 + \frac{1}{3}\sqrt{3}$, $\quad f''(x) = 6\left(2+\frac{1}{3}\sqrt{3}\right)-12 = 2\sqrt{3} > 0$

$\Rightarrow$ f is minimum at $x = 2 + \frac{1}{3}\sqrt{3}$

At $x = 2 - \frac{1}{3}\sqrt{3}$, $\quad f''(x) = 6\left(2+\frac{1}{3}\sqrt{3}\right) - 12 = -2\sqrt{3} < 0$

$\Rightarrow$ f is minimum at $x = 2 - \frac{1}{3}\sqrt{3}$.

Ex : Show that $x^5 - 5x^4 + 5x^3 - 1$ has a maximum when x = 1, a minimum when x = 3 and neither when x = 0.

Sol. Let $\quad f(x) = x^5 - 5x^4 + 5x^3 - 1$

$f'(x) = 5x^4 - 20x^3 + 15x^2$

$f''(x) = 20x^3 - 60x^2 + 30x$

For maximum or minimum.

$f'(x) = 0 \Rightarrow 5x^2(x^2 - 4x + 3) = 0$

$\Rightarrow \quad x^2(x - 1)(x - 3) = 0 \Rightarrow x = 0, 1, 3$

At x = 0, $\quad f''(x) = 0$

Since $\quad f'''(x) = 60x^2 - 120x + 30, f'''(0) = 30 \neq 0$

$\Rightarrow$ f has neither a max. nor mini. when x = 0

At x = 1, $\quad f''(x) = 20 - 60 + 30 = -10 < 0$

$\Rightarrow$ f has a maximum when x = 1

At x = 3, $\quad f''(x) = 20(3)^3 - 60(3)^2 + 30(3) = 90 > 0$

$\Rightarrow$ f has a minimum when = 3.

Ex : Examine the following function for extreme values : $(x - 3)^5 (x + 1)^4$.

Sol. Let $\quad f(x) = (x - 3)^5 (x + 1)^4$

then $\quad f'(x) = (x - 3)^5 . 4(x + 1)^3 + 5(x - 3)^4 (x + 1)^4$

$= (x - 3)^4 (x + 1)^3 [4(x - 3) + 5(x + 1)] = (x - 3)^4 (x + 1)^3 (9x - 7)$

For maximum or minimum,

$f'(x) = 0 \Rightarrow x = 3, -1, 7/9$

Let us test values one by one.

(i) For x slightly < 3, $\quad f'(x) = (+)(+)(+) = +ve$

For x slightly > 3, $\quad f'(x) = (+)(+)(+) = +ve$

Since f'(x) does not changes sign as a passes through 3, f is neither maximum nor minimum at x = 3.

(ii) For x slightly < –1, $f'(x) = (+)(-)(-) = +ve$

For x slightly > – 1, $f'(x) = (+)(+)(-) = -ve$

Since f'(x) changes sign from +ve to –ve as x passes through –1, f is maximum at x = –1

$f_{max} = f(-1) = 0$

(iii) For x slightly < $\frac{7}{9}$, $\quad f'(x) = (+)(+)(-) = -ve$

For x slightly > $\frac{7}{9}$, $f'(x) = (+)(+)(+) = +ve$

Since f'(x) changes sign from –ve to +ve as x passes through $\frac{7}{9}$, f is minimum at x = $\frac{7}{9}$

$$f_{min.} = f\left(\frac{7}{9}\right) = \left(\frac{7}{9}-3\right)^5\left(\frac{7}{9}+1\right)^4 = \frac{4^{13}.5^5}{3^{18}}.$$

Ex : Find the maximum and minimum values, if any, of the function $(1 - x)^2 e^x$.

Sol. Let $f(x) = (1 - x)^2 e^x$

$f'(x) = (1 - x)^2 . e^x - 2(1 - x)e^x$

$= (1 - x)(1 - x - 2) e^x = (1 - x)(-1 -x) e^x$

$= (x + 1)(x - 1) e^x = (x^2 - 1) e^x$

$f''(x) = (x^2 - 1).e^x + 2x . e^x = (x^2 + 2x - 1) e^x$

For maximum or minimum, f'(x) = 0

$\Rightarrow$ $(x^2 - 1)e^x = 0 \Rightarrow x^2 - 1 = 0$ $[\because e^x \neq 0 \text{ for any } x \in R]$

$\therefore$ $x = \pm 1$

When x = 1, $f''(x) = 2e > 0 \Rightarrow$ f is minimum at x = 1

$\therefore$ $f_{min.} = f(1) = 0$

When x = –1, $f''(x) = -2e^{-1} < 0 \Rightarrow$ f is maximum at x = –1

$f_{max.} = f(-1) = 4e^{-1} = 4/e.$

Ex : Find the maximum value of $\frac{\log x}{x}$, $0 < x < \infty$.

Sol. Let $f(x) = \frac{\log x}{x}$

then $f'(x) = \frac{x.\frac{1}{x} - \log x.1}{x^2} = \frac{1-\log x}{x^2}$

$$f''(x) = \frac{x^2\left(-\frac{1}{x}\right) - (1-\log x).2x}{x^4} = \frac{2x\log x - 3x}{x^4} = \frac{2\log x - 3}{x^3}$$

For maximum or min., f'(x) = 0 $\Rightarrow$ $1 - \log x = 0$

$\Rightarrow$ $\log x = 1 = \log e \Rightarrow x = e$

When x = e, $f''(x) = \frac{2\log e - 3}{e^3} = \frac{2-3}{e^3} = -\frac{1}{e^3} < 0$

$\Rightarrow$ f is maximum at x = e.

$$f_{max} = f(e)\frac{\log e}{e} = \frac{1}{e}.$$

Ex : Prove that the function $\left(\frac{1}{x}\right)^x$, x > 0 has a maximum at x = $\frac{1}{e}$.

For x slightly $> \frac{7}{9}$, $\quad f'(x) = (+)(+)(+) = +ve$

Since f'(x) changes sign from –ve to +ve as x passes through $\frac{7}{9}$, f is minimum at $x = \frac{7}{9}$

$$f_{min.} = f\left(\frac{7}{9}\right) = \left(\frac{7}{9} - 3\right)^5 \left(\frac{7}{9} + 1\right)^4 = \frac{4^{13}.5^5}{3^{18}}.$$

Ex : Find the maximum and minimum values, if any, of the function $(1 - x)^2 e^x$.

Sol. Let $\quad f(x) = (1 - x)^2 e^x$

$$f'(x) = (1 - x)^2 . e^x - 2(1 - x)e^x$$
$$= (1 - x)(1 - x - 2) e^x = (1 - x)(-1 - x) e^x$$
$$= (x + 1)(x - 1) e^x = (x^2 - 1) e^x$$
$$f''(x) = (x^2 - 1).e^x + 2x . e^x = (x^2 + 2x - 1) e^x$$

For maximum or minimum, $f'(x) = 0$

$\Rightarrow \quad (x^2 - 1)e^x = 0 \Rightarrow x^2 - 1 = 0 \quad [\because e^x \neq 0 \text{ for any } x \in R]$

$\therefore \quad x = \pm 1$

When x = 1, $f''(x) = 2e > 0 \Rightarrow$ f is minimum at x = 1

$\therefore \quad f_{min.} = f(1) = 0$

When x = –1, $f''(x) = -2e^{-1} < 0 \Rightarrow$ f is maximum at x = –1

$\quad f_{max.} = f(-1) = 4e^{-1} = 4/e.$

Ex : Find the maximum value of $\frac{\log x}{x}$, $0 < x < \infty$.

Sol. Let $\quad f(x) = \frac{\log x}{x}$

then $\quad f'(x) = \frac{x.\frac{1}{x} - \log x.1}{x^2} = \frac{1 - \log x}{x^2}$

$$f''(x) = \frac{x^2\left(-\frac{1}{x}\right) - (1 - \log x).2x}{x^4} = \frac{2x \log x - 3x}{x^4} = \frac{2 \log x - 3}{x^3}$$

For maximum or min., $f'(x) = 0 \quad \Rightarrow \quad 1 - \log x = 0$

$\Rightarrow \quad \log x = 1 = \log e \Rightarrow x = e$

When x = e, $\quad f''(x) = \frac{2\log e - 3}{e^3} = \frac{2 - 3}{e^3} = -\frac{1}{e^3} < 0$

$\Rightarrow$ f is maximum at x = e.

$$f_{max} = f(e)\frac{\log e}{e} = \frac{1}{e}.$$

Ex : Prove that the function $\left(\frac{1}{x}\right)^x$, $x > 0$ has a maximum at $x = \frac{1}{e}$.

Sol. Let $f(x) = \left(\frac{1}{x}\right)^x$ then $\log f(x) = x \log\frac{1}{x} = -x \log x$

$$\therefore \quad \frac{1}{f(x)}.f'(x) = -\left[x.\frac{1}{x} + 1.\log x\right] \Rightarrow f'(x) = -f(x)\ [1 + \log x]$$

$$\therefore \quad f''(x) = -f(x)\ .\ \frac{1}{x} - f'(x)\ .\ [1 + \log x]$$

For max. (or min.) $f'(x) = 0 \ \Rightarrow\ -f(x)\ [1 + \log x] = 0$

$\Rightarrow \quad 1 + \log x = 0 \qquad [\because\ f(x) \neq 0]$

$\Rightarrow \quad \log x = -1 \ \Rightarrow\ x = e^{-1}$

Also $\quad f''(e^{-1}) = -f(e^{-1})\ .\ e - f'(e^{-1})[1 + \log e^{-1}] = -(e)^{1/e}.\ e - 0 < 0$

$[\because\ f'(e^{-1}) = 0]$

$\Rightarrow$ f is maximum at $x = e^{-1} = \frac{1}{e}$.

Ex : Show that $\sin x(1 + \cos x)$ is a maximum when $x = \frac{\pi}{3}$.

Sol. Let $\quad f(x) = \sin x(1 + \cos x)$

then $\quad f'(x) = \sin x(-\sin x) + \cos x(1 + \cos x)$

$\qquad = \cos^2 x - \sin^2 x + \cos x = \cos 2x + \cos x$

$\quad f''(x) = -2 \sin 2x - \sin x$

For max. or min., $f'(x) = 0$

$$\Rightarrow \quad \cos 2x + \cos x = 0 \ \Rightarrow\ 2 \cos\frac{3x}{2}\cos\frac{x}{2} = 0$$

$$\Rightarrow \quad \text{either } \frac{3x}{2} = \frac{\pi}{2} \ \text{ or } \ \frac{x}{2} = \frac{\pi}{2} \ \Rightarrow\ x = \frac{\pi}{3} \ \text{ or } \ x = \pi.$$

Here we have to consider only the point $x = \frac{\pi}{3}$

$$f''\left(\frac{\pi}{3}\right) = -2\sin\frac{2\pi}{3} - \sin\frac{\pi}{3} = -2.\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} = -\frac{3\sqrt{3}}{2} < 0$$

$\Rightarrow$ f(x) has a maximum at $x = \frac{\pi}{3}$.

Ex : Find the maximum and minimum values of the function

$$\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x,\ 0 \le x \le \pi$$

Sol. Let $\quad f(x) = \sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x$

then $\quad f'(x) = \cos x + \cos 2x + \cos 3x$

$\quad f''(x) = -[\sin x + 2 \sin 2x + 3 \sin 3x]$

For max. or min., $f'(x) = 0$

$\Rightarrow \quad \cos x + \cos 2x + \cos 3x = 0 \quad \Rightarrow \quad (\cos 3x + \cos x) + \cos 2x = 0$

$\Rightarrow \quad 2\cos 2x \cos x + \cos 2x = 0 \quad \Rightarrow \quad \cos 2x(2\cos x + 1) = 0$

$\Rightarrow$ either $\cos 2x = 0$ or $\cos x = -\frac{1}{2}$

$\Rightarrow$ either $2x = \frac{\pi}{2}, \frac{3\pi}{2}$ or $x = \frac{2\pi}{3} \quad \Rightarrow \quad x = \frac{\pi}{4}, \frac{2\pi}{3}, \frac{3\pi}{4}$

When $x = \frac{\pi}{4}$, $f''(x) = -\left[\sin\frac{\pi}{4} + 2\sin\frac{\pi}{2} + 3\sin\frac{3\pi}{4}\right] = -\left[\frac{1}{\sqrt{2}} + 2 + \frac{3}{\sqrt{2}}\right] = -(2 + 2\sqrt{2}) < 0$

$\Rightarrow$ f is maximum at $x = \frac{\pi}{4}$

$$f_{max.} = f\left(\frac{\pi}{4}\right) = \sin\frac{\pi}{4} + \frac{1}{2}\sin\frac{\pi}{2} + \frac{1}{3}\sin\frac{3\pi}{4} = \frac{1}{\sqrt{2}} + \frac{1}{2} + \frac{1}{3}\cdot\frac{1}{\sqrt{2}} = \frac{4\sqrt{2}+3}{6}$$

When $x = \frac{2\pi}{3}$, $f''(x) = -\left[\sin\frac{2\pi}{3} + 2\sin\frac{4\pi}{3} + 3\sin 2\pi\right] = -\left[\frac{\sqrt{3}}{2} - 2.\frac{\sqrt{3}}{2} + 0\right] = \frac{\sqrt{3}}{2} > 0$

$\Rightarrow$ f is maximum at $x = \frac{2\pi}{3}$

$$f_{min.} = f\left(\frac{2\pi}{3}\right) = \sin\frac{2\pi}{3} + \frac{1}{2}\sin\frac{4\pi}{3} + \frac{1}{3}\sin 2\pi = \frac{\sqrt{3}}{2} + \frac{1}{2}\left(-\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{3}}{4}$$

When $x = \frac{3\pi}{4}$, $f''(x) = -\left[\sin\frac{3\pi}{4} + 2\sin\frac{3\pi}{2} + 3\sin\frac{9\pi}{4}\right]$

$$= -\left[\frac{1}{\sqrt{2}} - 2 + 3.\frac{1}{\sqrt{2}}\right] = -[2\sqrt{2} - 2] < 0$$

$\Rightarrow$ f is maximum at $x = \frac{3\pi}{4}$

$$f_{max} = f\left(\frac{3\pi}{4}\right) = \sin\frac{3\pi}{4} + \frac{1}{2}\sin\frac{3\pi}{4} + \frac{1}{3}\sin\frac{9\pi}{4}$$

$$= \frac{1}{\sqrt{2}} - \frac{1}{2} + \frac{1}{3}\cdot\frac{1}{\sqrt{2}} = \frac{4\sqrt{2}-3}{6}.$$

Ex : Prove that the function $f(\theta) = \sin^p \theta \cos^q \theta$ has a maximum at

$$q = \tan^{-1}\left(\sqrt{\frac{p}{q}}\right).$$

Sol. $f(\theta) = \sin^p \theta \cos^q \theta$

$f'(\theta) = \sin^p \theta \,.\, \theta \cos^{q-1} \theta (-\sin \theta) + p \sin^{p-1} \theta (\cos \theta) \cos^q \theta$

$= -q \sin^{p-1}\theta \cos^{q-1}\theta + p \sin^{p-1}\theta \cos^{q+1}\theta$

$= \sin^{p-1}\theta \cos^{q-1}\theta\,(p\cos^2\theta - q\sin^2\theta)$

For maxima or minima, $f'(\theta) = 0$

$\Rightarrow \quad \sin\theta = 0 \quad \text{or} \quad \cos\theta = 0 \quad \text{or} \quad p = \cos^2\theta = q\sin^2\theta$

$\Rightarrow \quad \theta = 0 \quad \text{or} \quad \theta = \dfrac{\pi}{2} \quad \text{or} \quad \tan\theta = \sqrt{\dfrac{p}{q}}$

Also $\quad f''(\theta) = \sin^{p-1}\theta\cos^{q-1}\theta\,(p\cos^2\theta - q\sin^2\theta)$

$$= \frac{\sin^p\theta\cos^q\theta}{\sin\theta\cos\theta}(p\cos^2\theta - q\sin^2\theta) = f(\theta)\,[p\cot\theta - q\tan\theta]$$

$\therefore \quad f''(\theta) = f(\theta)\,[-p\,\mathrm{cosec}^2\theta - q\sec^2\theta] + f'(\theta)\,(p\cot\theta - q\tan\theta)$

At $\theta = \tan^{-1}\sqrt{\dfrac{p}{q}}$, $f'(\theta) = 0$

$\therefore \quad f''(\theta) = -f(\theta)\,[p\,\mathrm{cosec}^2\theta + q\sec^2\theta]$ at $\theta = \tan^{-1}\sqrt{\dfrac{p}{q}}$ which is negative

Hence $f(\theta)$ is maximum at $\theta = \tan^{-1}\left(\sqrt{\dfrac{p}{q}}\right)$.

5. INDETERMINATE FORMS AND L'HOSPITAL'S RULE

Theorem : (L'Hospital's Rule) : Suppose f and g are differentiable and $g'(x) \neq 0$ near a (except possible at a). Suppose that

$$\lim_{x\to a} f(x) = 0 \quad \text{and} \quad \lim_{x\to a} g(x) = 0$$

or that

$$\lim_{x\to a} f(x) = +\infty \quad \text{and} \quad \lim_{x\to a} g(x) = \pm\infty$$

(In other words, we have an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$). Then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

if the limit on the right side exists (or is ∞ or $-\infty$).

Indeterminate Forms of Type $\frac{0}{0}$ and $\frac{\infty}{\infty}$

Ex. Find $\lim_{x\to\infty}\frac{5x-2}{7x+3}$

Sol. We have

$$\lim_{x\to\infty}\frac{5x-2}{7x+3} = \left[\frac{\infty}{\infty}\right] = \lim_{x\to\infty}\frac{(5x-2)'}{(7x+3)'} = \lim_{x\to\infty}\frac{(5x)'-2'}{(7x)'+3'} = \lim_{x\to\infty}\frac{5x'-2'}{7x'+3'} = \lim_{x\to\infty}\frac{5\cdot1-0}{7\cdot1+0} = \lim_{x\to\infty}\frac{5}{7} = \frac{5}{7}$$

In short,

$$\lim_{x\to\infty}\frac{5x-2}{7x+3} = \lim_{x\to\infty}\frac{(5x-2)'}{(7x+3)'} = \lim_{x\to\infty}\frac{5}{7} = \frac{5}{7}$$

Ex. Find $\lim_{x\to-2}\frac{x+2}{\ln(x+3)}$

Sol. We have

$$\lim_{x\to-2}\frac{x+2}{\ln(x+3)} = \left[\frac{0}{0}\right] = \lim_{x\to-2}\frac{(x+2)'}{(\ln(x+3))'} = \left\{\lim_{x\to-2}\frac{(x+2)'}{\frac{1}{x+3}\cdot(x+3)'} = \lim_{x\to-2}\frac{x'+2'}{\frac{1}{x+3}\cdot(x'+3')}\right.$$

$$\left. = \lim_{x\to-2}\frac{1+0}{\frac{1}{x+3}\cdot(1+0)}\right\} = \lim_{x\to-2}\frac{1}{\frac{1}{x+3}} = \lim_{x\to-2}\frac{1\cdot(x+3)}{\frac{1}{x+3}\cdot(x+3)} = \lim_{x\to-2}\frac{x+3}{1} = \lim_{x\to-2}(x+3) = -2+3 = 1$$

In short,

$$\lim_{x\to-2}\frac{x+2}{\ln(x+3)} = \lim_{x\to-2}\frac{(x+2)'}{(\ln(x+3))'} = \lim_{x\to-2}\frac{1}{(x+3)^{-1}} = \lim_{x\to-2}(x+3) = 1$$

Ex. Find $\lim\limits_{x\to\infty} \frac{3^x}{x^2+x-1}$.

Sol. We have

$$\lim_{x\to\infty} \frac{3^x}{x^2+x-1} = \left[\frac{\infty}{\infty}\right] = \lim_{x\to\infty} \frac{(3^x)'}{(x^2+x-1)'} = \lim_{x\to\infty} \frac{3^x \ln 3}{2x+1} = \left[\frac{\infty}{\infty}\right] = \lim_{x\to\infty} \frac{(3^x \ln 3)'}{(2x+1)'}$$

$$= \lim_{x\to\infty} \frac{\ln 3 (3^x)'}{2} = \lim_{x\to\infty} \frac{\ln 3 \cdot 3^x \cdot \ln 3}{2} = \infty$$

Ex. Find $\lim\limits_{x\to\infty} \frac{\ln x}{\sqrt{x}}$.

Sol. We have

$$\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}} = \left[\frac{\infty}{\infty}\right] = \lim_{x\to\infty} \frac{(\ln x)'}{(x^{1/2})'} = \lim_{x\to\infty} \frac{x^{-1}}{\frac{1}{2}x^{-1/2}} = \lim_{x\to\infty} \frac{x^{-1}\cdot x}{\frac{1}{2}x^{-1/2}\cdot x} = \lim_{x\to\infty} \frac{1}{\frac{1}{2}x^{1/2}} = 0$$

Ex. Find $\lim\limits_{x\to\infty} \frac{5x-\tan 5x}{x^3}$.

Sol. We have

$$\lim_{x\to\infty} \frac{5x-\tan 5x}{x^3} = \left[\frac{0}{0}\right] = \lim_{x\to\infty} \frac{(5x-\tan 5x)'}{(x^3)'} = \lim_{x\to 0} \frac{5-\sec^2 5x \cdot (5x)'}{3x^2} = \lim_{x\to 0} \frac{5-\sec^2 5x \cdot 5}{3x^2}$$

$$= \lim_{x\to 0} \frac{5(1-\sec^2 5x)}{3x^2} = \frac{5}{3}\lim_{x\to\infty} \frac{1-\sec^2 5x}{x^2}$$

Since $\lim\limits_{x\to 0}\frac{1-\sec^2 5x}{x^2}$ is an indeterminate form of type $\frac{0}{0}$, we can use L'Hospital's Rule again. But it is easier to do trigonometry instead. Note that $1 - \sec^2 5x = -\tan^2 5x$. Therefore

$$\frac{5}{3}\lim_{x\to 0}\frac{1-\sec^2 5x}{x^2} = \frac{5}{3}\lim_{x\to 0}\frac{-\tan^2 5x}{x^2} = -\frac{5}{3}\lim_{x\to 0}\frac{\tan^2 5x}{x^2} = -\frac{5}{3}\lim_{x\to 0}\frac{\frac{\sin^2 5x}{\cos^2 5x}}{x^2} = -\frac{5}{3}\lim_{x\to 0}\frac{\frac{\sin^2 5x}{1}}{x^2}$$

$$= -\frac{5}{3}\lim_{x\to 0}\frac{\sin^2 5x}{x^2} = \left[\frac{0}{0}\right] = -\frac{5}{3}\lim_{x\to 0}\left(\frac{\sin 5x}{x}\right)^2 = -\frac{5}{3}\left(\lim_{x\to 0}\frac{\sin 5x}{x}\right)^2 = -\frac{5}{3}\left(5\lim_{x\to 0}\frac{\sin 5x}{5x}\right)^2$$

$$= \left[\lim_{u\to 0}\frac{\sin u}{u} = 1\right] = -\frac{5}{3}(5\cdot 1)^2 = -\frac{5}{3}\cdot 25 = -\frac{125}{3}$$

Ex. Find $\lim\limits_{x\to\infty} \frac{x+\sin x}{x+\cos x}$

Sol. $$\lim_{x\to\infty} \frac{x+\sin x}{x+\cos x} = \left[\frac{\infty}{\infty}\right] = \lim_{x\to\infty} \frac{(x+\sin x)'}{(x+\cos x)'} = \lim_{x\to\infty} \frac{x'-(\sin x)'}{x'+(\cos x)'} = \lim_{x\to\infty} \frac{1+\cos x}{1-\sin x}$$

One can show, however, that $\lim\limits_{x\to\infty}\dfrac{1+\cos x}{1-\sin x}$ does not exist. In fact, we first note that 1 + cos x and 1 – sin x may attain any value between 0 and 2. From this one can deduce that $\dfrac{1+\cos x}{1-\sin x}$ attains any nonnegative value infinitely often as $x \to \infty$. This means that $\lim\limits_{x\to\infty}\dfrac{1+\cos x}{1-\sin x}$ does not exist, so L'Hospital's Rule can't be applied here.

Ex. Find $\lim\limits_{x\to\pi^-}\dfrac{\sin x}{1-\cos x}$.

Sol. We have

$$\lim_{x\to\pi^-}\frac{\sin x}{1-\cos x}=\lim_{x\to\pi^-}\frac{(\sin x)'}{(1-\cos x)'}=\lim_{x\to\pi^-}\frac{\cos x}{\sin x}=-\infty$$

The is **WRONG.** In fact, although the numerator sin x → 0 as $x \to \pi^-$, notice that the denominator (1 – cos x) does not approach 0, so L'Hosptial Rule can't be applied here. The required limit is easy to find, because the function is continuous at π and the denominator is nonzero here:

$$\lim_{x\to\pi^-}\frac{\sin x}{1-\cos x}=\frac{\sin\pi}{1-\cos\pi}=\frac{0}{1-(-1)}=0$$

Ex. Find $\lim\limits_{x\to 0}\dfrac{\sin x}{x}$.

Sol. We have

$$\lim_{x\to 0}\frac{\sin x}{x}=\left[\frac{0}{0}\right]=\lim_{x\to 0}\frac{(\sin x')}{x'}=\lim_{x\to 0}\frac{\cos x}{1}=\frac{\cos 0}{1}=\frac{1}{1}=1$$

Indeterminate Forms of Type ∞ – ∞ and 0 · ∞

Ex. Find $\lim\limits_{x\to\infty}(x-\ln x)$.

Sol. We have

$$\lim_{x\to\infty}(x-\ln x)=[\infty-\infty]=\lim_{x\to\infty}\left(x\cdot 1-x\cdot\frac{\ln x}{x}\right)=\lim_{x\to\infty}x\left(1-\frac{\ln x}{x}\right)$$

Note that

$$\lim_{x\to\infty}\frac{\ln x}{x}=\left[\frac{\infty}{\infty}\right]=\lim_{x\to\infty}\frac{(\ln x)'}{x'}=\lim_{x\to\infty}\frac{x^{-1}}{1}=\lim_{x\to\infty}\frac{1}{x}=0$$

therefore

$$\lim_{x\to\infty}(x-\ln x)=\lim_{x\to\infty}x\left(1-\frac{\ln x}{x}\right)=\lim_{x\to\infty}x(1-0)=\infty$$

OR We have

$$\lim_{x\to\infty}(x-\ln x)=[\infty-\infty]=\lim_{x\to\infty}(\ln(e)^x-\ln x)=\lim_{x\to\infty}\ln\left(\frac{e^x}{x}\right)$$

$$=\ln\left(\lim_{x\to\infty}\frac{e^x}{x}\right)=\left[\frac{\infty}{\infty}\right]=\ln\left(\lim_{x\to\infty}\frac{(e^x)'}{x}\right)=\ln\left(\lim_{x\to\infty}\frac{e^x}{1}\right)=\infty$$

Ex. Find $\lim_{x\to 1}\left(\frac{1}{\ln x}-\frac{1}{x-1}\right)$.

Sol. We have

$$\lim_{x\to 1}\left(\frac{1}{\ln x}-\frac{1}{x-1}\right)=[\infty-\infty]=\lim_{x\to 1}\left(\frac{1\cdot(x-1)}{\ln x\cdot(x-1)}-\frac{\ln x\cdot 1}{\ln x\cdot(x-1)}\right)$$

$$=\lim_{x\to 1}\frac{x-1-\ln x}{\ln x(x-1)}=\left[\frac{0}{0}\right]=\lim_{x\to 1}\frac{(x-1-\ln x)'}{(\ln x(x-1))'}$$

$$=\lim_{x\to 1}\frac{x'-1'-(\ln x)'}{(\ln x)'\cdot(x-1)+\ln x\cdot(x-1)'}=\lim_{x\to 1}\frac{1-\frac{1}{x}}{\frac{x-1}{x}+\ln x}$$

$$=\lim_{x\to 1}\frac{\left(1-\frac{1}{x}\right)x}{\left(\frac{x-1}{x}+\ln x\right)x}=\lim_{x\to 1}\frac{1\cdot x-\frac{1}{x}\cdot x}{\frac{x-1}{x}\cdot x+\ln x\cdot x}$$

$$=\lim_{x\to 1}\frac{x-1}{x-1+x\ln x}=\left[\frac{0}{0}\right]=\lim_{x\to 1}\frac{(x-1)'}{(x-1+x\ln x)'}=\lim_{x\to 1}\frac{x'-1'}{x'-1'+x'\ln x+x(\ln x)'}$$

$$=\lim_{x\to 1}\frac{1-0}{1-0+1\cdot\ln x+x\cdot\frac{1}{x}}=\lim_{x\to 1}\frac{1}{2+\ln x}=\frac{1}{2+0}=\frac{1}{2}$$

Ex. Find $\lim_{x\to\pi/2}(\tan x)^{2x-\pi}$.

Sol. Note that $\lim_{x\to\pi/2}(\tan x)^{2x-\pi}$ is ∞^0 type of an indeterminate form. Put

$$y=(\tan x)^{2x-\pi}$$

then

$\ln y=\ln((\tan x)^{2x-\pi})=(2x-\pi)\ln(\tan x)$

$$=\left\{\frac{(2x-\pi)\ln(\tan x)}{1}=\frac{(2x-\pi)^{-1}\cdot(2x-\pi)\ln(\tan x)}{(2x-\pi)^{-1}\cdot 1}\right\}=\frac{\ln(\tan x)}{(2x-\pi)^{-1}}$$

We have

$$\lim_{x\to\pi/2}\frac{\ln(\tan x)}{(2x-\pi)^{-1}}=\left[\frac{\infty}{\infty}\right]=\lim_{x\to\pi/2}\frac{[\ln(\tan x)]'}{[(2x-\pi)^{-1}]'}=\lim_{x\to\pi/2}\frac{\frac{1}{\tan x}\cdot(\tan x)'}{(-1)(2x-\pi)^{-2}\cdot(2x-\pi)'}$$

$$=\lim_{x\to\pi/2}\frac{\frac{1}{\tan x}\cdot\sec^2 x}{(-1)(2x-\pi)^{-2}\cdot 2}=-\frac{1}{2}\lim_{x\to\pi/2}\frac{\frac{\frac{1}{\sin x}\cdot\frac{1}{\cos^2 x}}{\cos x}}{(2x-\pi)^{-2}}=-\frac{1}{2}\lim_{x\to\pi/2}\frac{\frac{1}{\frac{\sin x}{\cos x}\cdot\cos^2 x}}{(2x-\pi)^{-2}}$$

$$=-\frac{1}{2}\lim_{x\to\pi/2}\frac{\frac{1}{\sin x\cos x}}{(2x-\pi)^{-1}}=-\frac{1}{2}\lim_{x\to\pi/2}\frac{\frac{1}{\sin x\cos x}\cdot\sin x\cos x(2x-\pi)^2}{(2x-\pi)^{-2}\cdot\sin x\cos x(2x-\pi)^2}=-\frac{1}{2}\lim_{x\to\pi/2}\frac{(2x-\pi)^2}{\sin x\cos x}$$

$$=-\frac{1}{2}\lim_{x\to\pi/2}\frac{(2x-\pi)^2}{\sin\left(\frac{\pi}{2}\right)\cos x}=-\frac{1}{2}\lim_{x\to\pi/2}\frac{(2x-\pi)^2}{\cos x}=\left[\frac{0}{0}\right]=-\frac{1}{2}\lim_{x\to\pi/2}\frac{[(2x-\pi)^2]'}{(\cos x)'}$$

$$=-\frac{1}{2}\lim_{x\to\pi/2}\frac{2(2x-\pi)\cdot(2x-\pi)'}{-\sin x}=-\frac{1}{2}\lim_{x\to\pi/2}\frac{2(2x-\pi)\cdot 2}{-\sin x}=-\frac{1}{2}\cdot\frac{2\cdot\left(2\cdot\frac{\pi}{2}-\pi\right)\cdot 2}{-\sin\frac{\pi}{2}}=-\frac{1}{2}\cdot\frac{2\cdot 0\cdot 2}{-1}=0$$

Therefore $\lim\limits_{x\to\pi/2}(\tan x)^{2x-\pi}=e^0=1$

Ex. Find $\lim\limits_{x\to 0^+} x^x$.

Sol. In short, $y = x^x \Rightarrow \ln y = x\ln x = \frac{\ln x}{x^{-1}}$

We have

$$\lim_{x\to 0^+}\frac{\ln x}{x^{-1}}=\lim_{x\to 0^+}\frac{(\ln x)'}{(x^{-1})'}=\lim_{x\to 0^+}\frac{x^{-1}}{-x^{-2}}=-\lim_{x\to 0^+}x=0$$

Therefore

$$\lim_{x\to 0^+}x^x=e^0=1$$

Ex. Find $\lim\limits_{x\to 0^+}(\tan 5x)^x$.

Sol. Note that $\lim\limits_{x\to 0^+}(\tan 5x)^x$ is 0^0 type of an indeterminate form. Put

$y = (\tan 5x)^x$

then

$$\ln y = \ln(\tan 5x)^x = x\ln(\tan 5x) = \left\{\frac{x\ln(\tan 5x)}{1}=\frac{x^{-1}\cdot x\ln(\tan 5x)}{x^{-1}\cdot 1}\right\}=\frac{\ln(\tan 5x)}{x^{-1}}$$

We have

$$\lim_{x\to 0^+}\frac{\ln(\tan 5x)}{x^{-1}}=\left[\frac{\infty}{\infty}\right]=\lim_{x\to 0^+}\frac{(\ln(\tan 5x))'}{(x^{-1})'}=\lim_{x\to 0^+}\frac{\frac{1}{\tan 5x}\cdot(\tan 5x)'}{-x^{-2}}$$

$$=\lim_{x\to 0^+}\frac{\frac{1}{\tan 5x}\cdot\sec^2 5x\cdot(5x)'}{-x^{-2}}=\lim_{x\to 0^+}\frac{\frac{1}{\tan 5x}\cdot\sec^2 5x\cdot 4}{-x^{-2}}$$

$$=\lim_{x\to 0^+}\frac{\frac{1}{\frac{\sin 5x}{\cos 5x}}\cdot\sec^2 5x\cdot 5}{-x^{-2}}=\lim_{x\to 0^+}\frac{\frac{1}{\frac{\sin 5x}{\cos 0}}\cdot\sec^2 0\cdot 5}{-x^{-2}}$$

$$= \lim_{x\to 0^+} \frac{\frac{1}{\sin 5x}\cdot 5}{-x^{-2}} = \lim_{x\to 0^+} \frac{\frac{1}{\sin 5x}\cdot 5\cdot x^2 \sin 5x}{-x^{-2}\cdot x^2 \sin 5x}$$

$$= \lim_{x\to 0^+} \frac{5x^2}{-\sin 5x} = \left[\frac{0}{0}\right] = \lim_{x\to 0^+} \frac{(5x^2)'}{(-\sin 5x)'}$$

$$= \lim_{x\to 0^+} \frac{10x}{-\cos 5x\cdot(5x)'} = \lim_{x\to 0^+} \frac{10x}{-\cos 5x\cdot 5} = \frac{10\cdot 0}{-\cos 0\cdot 5} = 0$$

Therefore

$$\lim_{x\to 0^+} (\tan 5x)^x = e^0 = 1$$

Ex. Find $\lim_{x\to 0^+} (\sin 2x)^{\tan 3x}$.

Sol. Note that $\lim_{x\to 0^+} (\sin 2x)^{\tan 3x}$ is 0^0 type of an indeterminate form. Put

$$y = (\sin 2x)^{\tan 3x}$$

then

$$\ln y = \ln((\sin 2x)^{\tan 3x}) = \tan 3x \ln(\sin 2x) = \frac{\sin 3x \ln(\sin 2x)}{\cos 3x}$$

We have

$$\lim_{x\to 0^+} \frac{\sin 3x \ln(\sin 2x)}{\cos 3x} = \frac{\lim_{x\to 0^+}(\sin 3x \ln(\sin 2x))}{\lim_{x\to 0^+} \cos 3x} = \lim_{x\to 0^+}(\sin 3x \ln(\sin 2x)) = [0\cdot\infty]$$

$$= \left\{\lim_{x\to 0^+} \frac{\sin 3x \ln(\sin 2x)}{1} = \lim_{x\to 0^+} \frac{(\sin 3x)^{-1}\cdot \sin 3x \ln(\sin 2x)}{(\sin 3x)^{-1}\cdot 1}\right\} = \lim_{x\to 0^+} \frac{\ln(\sin 2x)}{(\sin 3x)^{-1}} = \left[\frac{\infty}{\infty}\right]$$

$$= \lim_{x\to 0^+} \frac{[\ln(\sin 2x)]'}{((\sin 3x)^{-1})'} = \lim_{x\to 0^+} \frac{\sin^{-1} 2x\cdot(\sin 2x)'}{-(\sin 3x)^{-2}\cdot(\sin 3x)'} = \lim_{x\to 0^+} \frac{\sin^{-1} 2x\cdot \cos 2x\cdot(2x)'}{-(\sin 3x)^{-2}\cdot \cos 3x\cdot(3x)'}$$

$$= \lim_{x\to 0^+} \frac{\sin^{-1} 2x\cdot \cos 2x\cdot 2}{-(\sin 3x)^{-2}\cdot \cos 3x\cdot 3} = -\frac{2}{3}\lim_{x\to 0^+} \frac{\sin^{-1} 2x}{(\sin 3x)^{-2}}\cdot \lim_{x\to 0^+} \frac{\cos 2x}{\cos 3x} = -\frac{2}{3}\lim_{x\to 0^+} \frac{\sin^{-1} 2x}{(\sin 3x)^{-2}}$$

$$= -\frac{2}{3}\lim_{x\to 0^+} \frac{\sin^{-1} 2x\cdot \sin 2x\cdot \sin^2 3x}{(\sin 3x)^{-2}\cdot \sin 2x\cdot \sin^2 3x} = -\frac{2}{3}\lim_{x\to 0^+} \frac{\sin^2 3x}{\sin 2x} = \left[\frac{0}{0}\right] = -\frac{2}{3}\lim_{x\to 0^+} \frac{(\sin^2 3x)'}{(\sin 2x)'}$$

$$= -\frac{2}{3}\lim_{x\to 0^+} \frac{2\sin 3x\cdot(\sin 3x)'}{\cos 2x\cdot(2x)'} = -\frac{2}{3}\lim_{x\to 0^+} \frac{2\sin 3x\cdot \cos 3x\cdot(2x)'}{\cos 2x\cdot 2}$$

$$= -\frac{2}{3}\lim_{x\to 0^+} \frac{2\sin 3x\cdot \cos 3x\cdot 3}{\cos 2x\cdot 2} = -\frac{2}{3}\cdot\frac{2\sin 0\cdot \cos 0\cdot 3}{\cos 0\cdot 2} = -\frac{2}{3}\cdot\frac{2\cdot 0\cdot 1\cdot 3}{1\cdot 2} = 0$$

Therefore $\lim_{z\to 0^+} (\sin 2x)^{\tan 3x} = e^0 = 1$

Ex. Find $\lim\limits_{x\to\infty}\left(\frac{x+1}{x+2}\right)^{x}$

Sol. Note that $\lim\limits_{x\to\infty}\left(\frac{x+1}{x+2}\right)^{x}$ is 1^{∞} type of an indeterminate form. Put $y = \left(\frac{x+1}{x+2}\right)^{-x}$,

then

$$\ln y = \ln\left(\left(\frac{x+1}{x+2}\right)^{x}\right) = x\ln\left(\frac{x+1}{x+2}\right) = \left\{\frac{x\ln\left(\frac{x+1}{x+2}\right)}{1} = \frac{x^{-1}\cdot x\ln\left(\frac{x+1}{x+2}\right)}{x^{-1}\cdot 1}\right\} = \frac{\ln\left(\frac{x+1}{x+2}\right)}{x^{-1}}$$

We have

$$\lim_{x\to\infty}\frac{\ln\left(\frac{x+1}{x+2}\right)}{x^{-1}} = \left[\frac{0}{0}\right] = \lim_{x\to\infty}\frac{\left(\ln\left(\frac{x+1}{x+2}\right)\right)'}{(x^{-1})'} = \lim_{x\to\infty}\frac{\frac{1}{\frac{x+1}{x+2}}\cdot\left(\frac{x+1}{x+2}\right)'}{-x^{-2}}$$

$$= \lim_{x\to\infty}\frac{\frac{x+2}{x+1}\cdot\frac{(x+1)'(x+2)-(x+1)(x+2)'}{(x+2)'}}{-x^{-2}} = \lim_{x\to\infty}\frac{\frac{x+2}{x+1}\cdot\frac{1\cdot(x+2)-1\cdot(x+1)}{(x+2)^2}}{-x^{-2}}$$

$$= \lim_{x\to\infty}\frac{\frac{x+2}{x+1}\cdot\frac{x+2-x-1}{(x+2)^2}}{-x^{-2}} = \lim_{x\to\infty}\frac{\frac{x+2}{x+1}\cdot\frac{1}{(x+2)^2}}{-x^{-2}} = \lim_{x\to\infty}\frac{\frac{1}{(x+1)(x+2)}}{-x^{2}}$$

$$= \lim_{x\to\infty}\frac{\frac{1}{(x+1)(x+2)}\cdot x^2(x+1)(x+2)}{-x^{-2}\cdot x^2(x+1)(x+2)} = \lim_{x\to\infty}\frac{x^2}{-(x+1)(x+2)} = \left[\frac{\infty}{\infty}\right] = -\lim_{x\to\infty}\frac{x^2}{x^2+3x+1}$$

$$= \lim_{x\to\infty}\frac{\frac{x^2}{x^2}}{\frac{x^2+3x+1}{x^2}} = -\lim_{x\to\infty}\frac{\frac{x^2}{x^2}}{\frac{x^2}{x^2}+\frac{3x}{x^2}+\frac{1}{x^2}} = -\lim_{x\to\infty}\frac{1}{1+\frac{3}{x}+\frac{1}{x^2}} = -\frac{1}{1+0+0} = -1$$

Therefore, $= \lim\limits_{x\to\infty}\left(\frac{x+1}{x+2}\right)^{x} = e^{-1}$

Ex. Find $\lim\limits_{x\to 0^+}(1 + \sin 7x)^{\cot 5x}$.

Sol. Note that $\lim\limits_{x\to 0^+}(1 + \sin 7x)^{\cot 5x}$ is 1^{∞} type of an indeterminate form. Put

$$y = (1 + \sin 7x)^{\cot 5x}$$

then

$$\ln y = \ln((1 + \sin 7x)^{\cot 5x}) = \cot 5x \ln(1 + \sin 7x) = \frac{\cos 5x\ln(1+\sin 7x)}{\sin 5x}$$

We have

$$\lim_{x\to 0^+}\frac{\cos 5x\ \ln(1+\sin 7x)}{\sin 5x}=\lim_{x\to 0^+}\cos 5x\cdot\lim_{x\to 0^+}\frac{\ln(1+\sin 7x)}{\sin 5x}$$

$$=\lim_{x\to 0^+}\frac{\ln(1+\sin 7x)}{\sin 5x}=\left[\frac{0}{0}\right]=\lim_{x\to 0^+}\frac{[\ln(1+\sin 7x)]'}{(\sin 5x)'}=\lim_{x\to 0^+}\frac{\frac{1}{1+\sin 7x}\cdot(1+\sin 7x)'}{\cos 5x\cdot(5x)'}$$

$$=\lim_{x\to 0^+}\frac{\frac{1}{1+\sin 7x}\cdot\cos 7x\cdot(7x)'}{\cos 5x\cdot 5}=\lim_{x\to 0^+}\frac{\frac{1}{1+\sin 7x}\cdot\cos 7x\cdot 7}{\cos 5x\cdot 5}=\frac{\frac{1}{1+0}\cdot 1\cdot 7}{1\cdot 5}=\frac{7}{5}$$

Therefore $\lim_{x\to 0^+}(1+\sin 7x)^{\cot 5x}=e^{7/5}$